

Constructive and analytic enumeration of circulant graphs with p^3 vertices; $p = 3, 5$

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Abstract

Two methods, structural (constructive) and multiplier (analytical), of exact enumeration of undirected and directed circulant graphs of orders 27 and 125 are elaborated and represented in detail here together with intermediate and final numerical data. The first method is based on the known useful classification of circulant graphs in terms of S -rings and results in exhaustive listing (with the use of COCO and GAP) of all corresponding S -rings of the indicated orders. The latter method is conducted in the framework of a general approach developed earlier for

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counting circulant graphs of prime-power orders. This approach is based on a familiar isomorphism criterion for circulant graphs of such orders and on its subsequent adaptation to the enumeration in the Redfield–Pólya style. In particular, five intermediate enumeration subproblems arise, which are refined further into eleven subproblems of this type (5 and 11 are not accidentally the 3d Catalan and 3d little Schröder numbers, resp.). All of them are resolved for the four cases under consideration (again with the use of GAP). Except for the case of undirected circulant graphs of orders 27, the numerical results obtained here are new. In particular the number (up to isomorphism) of directed circulant graphs of orders 27, regardless of valency, is shown to be equal to 3,728,891. Some curious and rather unexpected identities are established between intermediate valency-specified enumerators (both for undirected and directed circulant graphs) and their validity is conjectured for arbitrary odd prime p . Two of them are similar to two new identities for prime-squared circulant graphs established and proved in Appendix A. In general this research can be considered as, presumably, the crucial input towards the explicit uniform enumeration formulae for circulant graphs of orders p^3 for arbitrary prime $p > 2$.

Keywords: circulant graph; cyclic group; S -ring; constructive enumeration; multiplier; enumeration under group action; graph isomorphism; Pólya’s method

Mathematics Subject Classifications: 05C30, 05C25, 05C20.

1 Introduction.

The present research is carried out in the framework of the general program outlined in the paper [5] for counting circulant graphs of prime-power orders. We refer to this paper for details concerning two approaches to the exact enumeration of circulant graphs, namely, constructive and analytical. Recall that the former is based on the known useful classification of circulant graphs in terms of S -rings. This not only counts the nonisomorphic circulant graphs but enables us, in principle at least, actually to list them.

The analytic approach is based on the familiar isomorphism theorem [7] for circulant graphs of prime-power orders. In analytic enumeration we are guided also by the subsequent adaptation of this theorem to the enumeration

of circulant graphs as developed in [10]. In particular, for circulant graphs of order p^3 their analytical (that is, formula-wise) enumeration has been reduced in this paper to five well-specified (and rather sophisticated) enumeration subproblems of Redfield–Pólya type. In order to obtain the solutions we refined them further into more elementary eleven subproblems.

From the very beginning we restrict ourselves to orders 27 and 125 only. There are several arguments for our choice. First of all, there are almost no numerical results for the number of isomorphism classes of circulant graphs of prime-cubed orders, including $p = 3$ and 5. There exist huge numbers of circulant graphs even of these orders so that it hardly makes much sense to enumerate constructively circulant graphs of larger orders. No such obstacle arises for analytical enumeration, but even here these two orders require much effort. Presumably, the main difficulties of analytical enumeration should become apparent already on these least prime-cubed orders. Moreover our aim is also to compare both approaches on the same classes of objects and to obtain confirmation of numerical results obtained in both ways using COCO and GAP and also partially by brute force.

As in several enumeration problems, in order to count the number of non-isomorphic structures of a certain type one needs to have a criterion for determining when two such structures are “the same”, in our case, are isomorphic. The importance of counting circulant graphs stems from the falsity of a very natural conjecture of Ádám giving a condition on the connecting sets for isomorphism to hold, which however turned out to be false: Ádám’s condition is sufficient for isomorphism but not necessary. The falsity of Ádám’s conjecture led to some beautiful results which characterised completely the conditions on the order of the circulant graph for Ádám’s Conjecture to hold. This led to two very important threads of research in algebraic graph theory: discovering what necessary and sufficient conditions on the connecting sets give isomorphism when Ádám’s Conjecture fails, and the question of when it is possible to determine the isomorphism of general Cayley graphs from conditions on the connecting sets.

In this paper, since we are enumerating circulant graphs, we shall use results from the first line of research, which we shall describe below. We shall adopt two very different methods which have been used for enumerating

non-isomorphic circulant graphs: the structural and the multiplier approach. We first introduce these approaches in the following subsections. Then, in the next sections, we shall present some results obtained for enumeration of circulant graphs of order p^k mainly for $p = 3, 5$ and $k = 2, 3$ using these methods, giving more detail for the multiplier approach. The results we present for $p = 5$ are new as are the results for directed circulants of order 3^3 . Both our numerical results and the generating functions which we obtain are important. In fact we point out several relations which arise between the intermediate terms which form these generating functions. We conclude by conjecturing that the relations which emerge from our generating functions for $k = 3$ and $p = 3, 5$ hold for all odd prime p . In an appendix we give some theoretical support for these conjectures by proving some similar relations for $k = 2$ for all odd prime p .

1.1 First definitions

A circulant graph is a Cayley graph of a cyclic group. That is, let G be a cyclic group (which we shall represent as the group \mathbb{Z}_n of integers with addition modulo n , the size of the group) and let $S \subseteq G$ (called the *connecting set* of the Cayley graph) such that $0 \notin S$. Then a circulant is a Cayley graph $\text{Cay}(G, S)$ which has G as vertex-set and two vertices g, h are adjacent if $g = h + s$ for some $s \in S$. We shall always assume that S generates G therefore the circulant graph $\text{Cay}(G, S)$ is connected. In the special case when $-S = S$ (that is, $s \in S$ if and only if $-s \in S$), the circulant graph is also referred to as an *undirected graph*. For brevity we shall sometimes refer to “circulants” instead of “circulant graphs”.

The *valency* of a vertex in an undirected graph is equal to the number of edges containing that vertex. The valency of a vertex v in a directed graph is equal to the number of arcs of the form (v, x) . In our generating functions, we usually denote valency by the letter r . Note that an edge $\{a, b\}$ is considered to be the union of the two arcs (a, b) and (b, a) , and this is consistent with our definitions of valency for directed and undirected graphs. Note also that our undirected graphs are therefore special cases of directed graphs in which every arc is accompanied by its opposite, and directed graphs could also be “mixed”, in the sense that they could contain both arcs and edges.

1.2 The structural approach: an introduction

The group ring $\langle \mathbb{Z}[\mathbb{Z}_n]; +, \cdot \rangle$ of \mathbb{Z}_n over \mathbb{Z} , consists of the set of all formal linear combinations of elements of \mathbb{Z}_n with integral coefficients, that is, all formal sums $\sum_{h \in \mathbb{Z}_n} \alpha_h \underline{h}$ with $\alpha_h \in \mathbb{Z}, h \in \mathbb{Z}_n$, together with addition

$$\sum_{h \in \mathbb{Z}_n} \alpha_h \underline{h} + \sum_{h \in \mathbb{Z}_n} \beta_h \underline{h} := \sum_{h \in \mathbb{Z}_n} (\alpha_h + \beta_h) \underline{h}$$

and formal multiplication

$$\left(\sum_{h \in \mathbb{Z}_n} \alpha_h \underline{h} \right) \cdot \left(\sum_{k \in \mathbb{Z}_n} \beta_k \underline{k} \right) := \sum_{h, k \in \mathbb{Z}_n} \alpha_h \beta_k \underline{(h+k)} = \sum_{h \in \mathbb{Z}_n} \left(\sum_{k \in \mathbb{Z}_n} \alpha_{h-k} \beta_k \right) \underline{h}.$$

Note that we are writing \underline{h} for $h \in \mathbb{Z}_n$ in order to distinguish clearly between elements of \mathbb{Z}_n and \mathbb{Z} .

The elements of $\mathbb{Z}[\mathbb{Z}_n]$ also satisfy the *Schur-Hadamard product* defined as follows

$$\left(\sum_{h \in \mathbb{Z}_n} \alpha_h \underline{h} \right) \circ \left(\sum_{h \in \mathbb{Z}_n} \beta_h \underline{h} \right) := \sum_{h \in \mathbb{Z}_n} (\alpha_h \beta_h) \underline{h}$$

Therefore, for $T, T' \subseteq \mathbb{Z}_n$ we have $\underline{T} \circ \underline{T'} = \underline{T \cap T'}$.

The \mathbb{Z} -submodule of $\mathbb{Z}[\mathbb{Z}_n]$ generated by elements $\lambda_1, \dots, \lambda_r \in \mathbb{Z}[\mathbb{Z}_n]$ will be denoted by

$$\langle \lambda_1, \dots, \lambda_r \rangle.$$

Therefore the \mathbb{Z} -submodule $\langle \lambda_1, \dots, \lambda_r \rangle$, consists of all linear combinations of $\lambda_1, \dots, \lambda_r$ and their products.

Assume $T \subseteq \mathbb{Z}_n, T = \{t_1, t_2, \dots, t_r\}$. Elements of the form

$$\underline{T} := \sum_{h \in T} \underline{h}$$

are called *simple quantities* of $\mathbb{Z}[\mathbb{Z}_n]$. One can consider \underline{T} as the formal sum $\sum_{h \in \mathbb{Z}_n} \alpha_h \underline{h}$ with $\alpha_h = 1$ if and only if $h \in T$ and $\alpha_h = 0$ otherwise, that is, a simple quantity is a list in which every entry has multiplicity 1. For

$T = \{t_1, t_2, \dots, t_r\}$ we use the notation

$$\underline{t_1, \dots, t_r}$$

instead of $\{t_1, \dots, t_r\}$.

A subring \mathfrak{S} of a group ring $\mathbb{Z}[\mathbb{Z}_n]$ is called a *Schur ring* \mathfrak{S} or \mathfrak{S} -ring over \mathbb{Z}_n , of rank r if the following conditions hold:

1. \mathfrak{S} is closed under addition and multiplication with elements from \mathbb{Z} (i.e. \mathfrak{S} is a \mathbb{Z} -module);
2. Simple quantities $\underline{T}_0, \underline{T}_1, \dots, \underline{T}_{r-1}$ exist in \mathfrak{S} such that every element $\sigma \in \mathfrak{S}$ has a unique representation;

$$\sigma = \sum_{i=0}^{r-1} \sigma_i \underline{T}_i$$

3. $\underline{T}_0 = \underline{0}$, $\sum_{i=0}^{r-1} \underline{T}_i = \underline{\mathbb{Z}_n}$, that is, $\{T_0, T_1, \dots, T_{r-1}\}$ is a partition of \mathbb{Z}_n ;
4. For every $i \in \{0, 1, 2, \dots, r-1\}$ there exists a $j \in \{0, 1, 2, \dots, r-1\}$ such that $\underline{T}_j = \underline{-T_i} (= \underline{\{n - x : x \in T_i\}})$ (therefore, $\underline{T_i}^t = \underline{T_j}$);
5. For $i, j \in \{1, \dots, r\}$, there exist non-negative integers p_{ij}^k called structure constants, such that

$$\underline{T}_i \cdot \underline{T}_j = \sum_{k=1}^r p_{ij}^k \underline{T}_k$$

The simple quantities $\underline{T}_0, \underline{T}_1, \dots, \underline{T}_{r-1}$ form a standard basis for \mathfrak{S} and their corresponding sets T_i are *basic sets* of the \mathfrak{S} -ring. The circulant graphs $\Gamma_i = \text{Cay}(\mathbb{Z}_n, T_i)$, where $0 \leq i \leq r-1$, are called *basic circulant graphs* [8]. The following notation will denote a \mathfrak{S} -ring generated by its basic sets $\underline{T}_0, \underline{T}_1, \dots, \underline{T}_{r-1}$:

$$\mathfrak{S} = \langle \underline{T}_0, \underline{T}_1, \dots, \underline{T}_{r-1} \rangle.$$

Note that both $\mathbb{Z}(\mathbb{Z}_n)$ and $\langle \underline{0}, \underline{\mathbb{Z}_n - \{0\}} \rangle$ are Schur rings over \mathbb{Z}_n which we call the trivial Schur rings over \mathbb{Z}_n .

A permutation $g : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ is called an automorphism of an \mathfrak{S} -ring \mathfrak{S} , if it is an automorphism of every graph Γ_i . Equivalently, the intersection

of the automorphism groups of the basic circulant graphs of an \mathfrak{S} -ring $\mathfrak{S} = \langle \underline{T}_0, \underline{T}_1, \dots, \underline{T}_{r-1} \rangle$, gives the automorphism group of the \mathfrak{S} -ring.

$$Aut\mathfrak{S} := \bigcap_{i=0}^{r-1} Aut\Gamma_i \quad (1.1)$$

The structural approach to the enumeration of circulants on n vertices is based on the lattice $\mathcal{L}(n)$ of all Schur rings over \mathbb{Z}_n which, together with information on the automorphism groups of the Schur rings, suffices to carry out the enumeration. This enumeration scheme has already been described in [5], so we give here only a brief summary.

We first use the lattice of Schur rings to count the number of labelled circulant graphs, as follows.

1. Construct the lattice $\mathcal{L}(n)$ of all Schur rings as a sequence $\mathcal{L}(n) = (\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_s)$ such that $\mathfrak{S}_j \subseteq \mathfrak{S}_i$ implies $j \leq i$;
2. For directed circulants, let \tilde{d}_{ir} be the number of r -element basis sets of the \mathfrak{S} -ring \mathfrak{S}_i , different from the basis set $T_0 = \{0\}$, that is,

$$\tilde{d}_{ir} := |\{T_{(x)} \in \mathfrak{S}_i \mid x \neq 0 \text{ and } |T_{(x)}| = r\}|$$

3. For undirected circulants, let d_{ir} be the number of r -element symmetrized (that is closed under taking of inverses) basis sets of \mathfrak{S}_i , different from T_0 . That is,

$$d_{ir} := |\{T_{(x)}^{sym} \mid x \neq 0 \text{ and } |T_{(x)}^{sym}| = r\}|$$

4. Enumeration of all labelled directed and undirected circulant graphs which belong to the Schur ring \mathfrak{S}_i may then be carried out by making use of generating functions $\tilde{f}_i(t)$ and $f_i(t)$ respectively, given by:

$$\begin{aligned} \tilde{f}_i(t) &:= \sum_{r=0}^{n-1} \tilde{f}_{ir} t^r := \prod_{r=1}^{n-1} (1 + t^r)^{\tilde{d}_{ir}} \\ f_i(t) &:= \sum_{r=0}^{n-1} f_{ir} t^r := \prod_{r=1}^{n-1} (1 + t^r)^{d_{ir}} \end{aligned} \quad (1.2)$$

Substituting $t = 1$ in the generating functions, would give us the number of all labelled directed and undirected circulant graphs in \mathfrak{S}_i . In addition, the graph corresponding to $T \in \mathfrak{S}_i$ is of valency r if T has r elements.

The link between the number of labelled and unlabelled circulant graphs is given by this result.

Lemma 1. [5] *Let $G_i = \text{Aut}(\mathfrak{S}_i)$, let $N(G_i) = N_{S_n}(G_i)$ be the normalizer of the group G_i in S_n , and let Γ be a circulant graph belonging to \mathfrak{S}_i . Then*

- (a) $\text{Aut}(\Gamma) = G_i \iff \Gamma$ generates \mathfrak{S}_i .
- (b) *If $\text{Aut}(\Gamma) = G_i$ then there are exactly $[N(G_i) : G_i]$ (that is, equal to the number of cosets of G_i in $N(G_i)$) distinct circulant graphs which are isomorphic to Γ .*

So, let the generating function for the number of non-isomorphic undirected circulant graphs with automorphism group G_i be given by

$$g_i(t) = \sum_{r=0}^{n-1} g_{ir} t^r$$

and let the generating function for the number of non-isomorphic directed circulant graphs with automorphism group G_i be given by

$$\tilde{g}_i(t) = \sum_{r=0}^{n-1} \tilde{g}_{ir} t^r$$

(In all our generating functions, the coefficient of t^r equals the number of circulants under consideration in which all vertices have valency r .)

Moreover, let

$$g(t) = g(n, t) \text{ and } \tilde{g}(t) = \tilde{g}(n, t)$$

denote the generating functions for the number of non-isomorphic undirected and directed circulant graphs, respectively, with n vertices. The values $g(1)$ and $\tilde{g}(1)$ therefore give the numbers of all non-isomorphic undirected and directed circulant graphs, respectively, with n vertices. These generating

functions are then given by the following theorem whose proof is based on the inclusion-exclusion principle.

Theorem 1 ([5]).

$$\begin{aligned} g_i(t) &= \frac{|G_i|}{|N(G_i)|} \left(f_i(t) - \sum_{\mathfrak{S}_j \subseteq \mathfrak{S}_i} \frac{|N(G_j)|}{|G_j|} g_j(t) \right), \\ \tilde{g}_i(t) &= \frac{|G_i|}{|N(G_i)|} \left(\tilde{f}_i(t) - \sum_{\mathfrak{S}_j \subseteq \mathfrak{S}_i} \frac{|N(G_j)|}{|G_j|} \tilde{g}_j(t) \right), \\ g(t) &= \sum_{i=1}^s g_i(t), \quad \tilde{g}(t) = \sum_{i=1}^s \tilde{g}_i(t). \end{aligned} \tag{1.3}$$

In Section 2 we shall give a few simple examples of this approach towards the enumeration of circulant graphs.

1.3 The multiplier approach: an introduction

It is clear that if $\Gamma_1 = \text{Cay}(G, S)$ and $\Gamma_2 = \text{Cay}(G, T)$ such that there exists an $m \in G$ with $mS = \{ms : s \in S\} = T$, then Γ_1 and Γ_2 are isomorphic. In this case we say that the connecting sets are *equivalent*. In [1] Ádám conjectured that the converse is also true, that is, two isomorphic circulant graphs have equivalent connecting sets. This conjecture turned out to be false. The following is the smallest counterexample, found by Elspas and Turner [2]. It is a pair of directed circulants. Let G be the group of integers modulo 8, $S = \{1, 2, 5\}$ and $T = \{1, 5, 6\}$, and let Γ_1, Γ_2 be the corresponding circulant graphs $\text{Cay}(G, S)$ and $\text{Cay}(G, T)$. Then the sets S, T are not equivalent but Γ_1, Γ_2 are isomorphic via the map

$$i \mapsto 4 \left\lfloor \frac{i+1}{2} \right\rfloor + i.$$

Further counterexamples with undirected circulants were also subsequently found.

The principal theorem which gives the most correct version of Ádám's Conjecture is the following due to Muzychuk [11].

Theorem 2. *Let Γ_1 and Γ_2 be two circulant graphs on n vertices, and sup-*

pose that n is square-free. Then Γ_1, Γ_2 are isomorphic if and only if their connecting sets are equivalent.

In any enumeration problem, determining when two objects are “isomorphic” is an essential step. Muzychuk’s Theorem therefore divides the problem into two classes: when n is square free and when n has repeated prime factors. The easiest square-free case occurs when n is prime, and the fact that, in this case, Ádám’s Conjecture holds, was first proved by Elspas and Turner [2]. This reduced the problem of enumerating circulant graphs on a prime number of vertices to that of determining the number of subsets of \mathbb{Z}_p^* which are not similar under the regular action of the multiplicative group \mathbb{Z}_p^* on itself. (Here, \mathbb{Z}_n^* denotes the group of units in \mathbb{Z}_n which, for $n = p$, prime, is equal to $\mathbb{Z}_p - 0$.) Elspas and Turner used this method to count the number of directed and undirected circulants on p vertices by means of a clever use of Pólya’s enumeration theorem.

In view of this result and Muzychuk’s Theorem, the natural non-square-free cases to consider would be when the order n is a power k of a prime, that is, $n = p^k$, for $k \geq 2$. But to enumerate circulant graphs of such an order requires some multiplicative relations between the connecting sets of two circulant graphs which are necessary and sufficient for them to be isomorphic, that is, we require the correct version of Ádám’s Conjecture for $n = p^k$. We call this method of enumerating non-isomorphic circulants the multiplier approach. We shall consider in some detail the multiplier approach for $k = 2, 3$ and $p = 3, 5$ in Section 3.

2 The structural approach

2.1 The case p^2 for $p = 3$

We shall first describe the structural approach for p^2 with $p = 3$. This work has already been shown in [5] but we present it here in order to illustrate the method. Using the techniques of wreath products and wreath decomposition of Schur rings as described in [5] one obtains that the following is the list of all Schur rings over \mathbb{Z}_9

$$\begin{aligned}
\mathfrak{S}_1 &= \langle \underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8} \rangle, \\
\mathfrak{S}_2 &= \langle \underline{0}, \underline{1}, \underline{2}, \underline{4}, \underline{5}, \underline{7}, \underline{8}, \underline{3}, \underline{6} \rangle, \\
\mathfrak{S}_3 &= \langle \underline{0}, \underline{1}, \underline{4}, \underline{7}, \underline{2}, \underline{5}, \underline{8}, \underline{3}, \underline{6} \rangle, \\
\mathfrak{S}_4 &= \langle \underline{0}, \underline{1}, \underline{2}, \underline{4}, \underline{5}, \underline{7}, \underline{8}, \underline{3}, \underline{6} \rangle, \\
\mathfrak{S}_5 &= \langle \underline{0}, \underline{1}, \underline{4}, \underline{7}, \underline{2}, \underline{5}, \underline{8}, \underline{3}, \underline{6} \rangle, \\
\mathfrak{S}_6 &= \langle \underline{0}, \underline{1}, \underline{8}, \underline{2}, \underline{7}, \underline{3}, \underline{6}, \underline{4}, \underline{5} \rangle, \\
\mathfrak{S}_7 &= \langle \underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8} \rangle,
\end{aligned}$$

We now show how this list can be used to enumerate all circulant graphs of order 9.

We give only the briefest necessary information about the automorphism groups of all S-rings in \mathcal{L} . These were obtained using GAP (see [12]), the automorphism groups being the intersection of the automorphism groups of the basic Cayley graphs associated with each Schur-ring.

Automorphism Group	Normalizer
$G_1 = S_9$,	$[N(G_1) : G_1] = 1$,
$G_2 = S_3 \wr S_3$,	$[N(G_2) : G_2] = 1$,
$G_3 = \mathbb{Z}_3 \wr S_3$,	$[N(G_3) : G_3] = 2$,
$G_4 = S_3 \wr \mathbb{Z}_3$,	$[N(G_4) : G_4] = 2$,
$G_5 = \mathbb{Z}_3 \wr \mathbb{Z}_3$,	$[N(G_5) : G_5] = 4$,
$G_6 = D_9$,	$[N(G_6) : G_6] = 3$,
$G_7 = \mathbb{Z}_9$,	$[N(G_7) : G_7] = 6$.

Now we are able to use the structural approach in order to count the number of undirected and directed circulant graphs of order 9.

$$\begin{aligned}
f_1(1) &= & \tilde{f}_1(1) &= 2, \\
f_2(1) &= & \tilde{f}_2(1) &= 2^2, \\
f_3(1) &= 2^2, & \tilde{f}_3(1) &= 2^3, \\
f_4(1) &= 2^2, & \tilde{f}_4(1) &= 2^3, \\
f_5(1) &= 2^2, & \tilde{f}_5(1) &= 2^4, \\
f_6(1) &= & \tilde{f}_6(1) &= 2^4, \\
f_7(1) &= 2^4, & \tilde{f}_7(1) &= 2^8.
\end{aligned}$$

Therefore

$$\begin{aligned}
g_1(1) &= 2, & \tilde{g}_1(1) &= 2, \\
g_2(1) &= 2^2 - 2 = 2, & \tilde{g}_2(1) &= 2, \\
g_3(1) &= \frac{1}{2}(2^2 - 2 - 2)0, & \tilde{g}_3(1) &= \frac{1}{2}(2^3 - 2 - 2) = 2, \\
g_4(1) &= \frac{1}{2}(2^2 - 2 - 2) = 0, & \tilde{g}_4(1) &= \frac{1}{2}(2^3 - 2 - 2) = 2, \\
g_5(1) &= \frac{1}{4}(2^2 - 2 - 2) = 0, & \tilde{g}_5(1) &= \frac{1}{4}(2^4 - 2 - 2 - 4 - 4) = 1, \\
g_6(1) &= \frac{1}{3}(2^4 - 2 - 2) = 4, & \tilde{g}_6(1) &= 4, \\
g_7(1) &= \frac{1}{6}(2^4 - 2 - 2 - 3 \cdot 4) = 0, & \tilde{g}_7(1) &= \frac{1}{6}(2^8 - 2 - 2 - 4 - 4 - 4 - 12) = 38.
\end{aligned}$$

$$g(1) = g(9, 1) = 8, \quad \tilde{g}(1) = \tilde{g}(9, 1) = 51,$$

2.2 The case p^3 for $p = 3$

We shall now use the structural approach to enumerate the *undirected* circulant graphs of order 27. The number of such circulant graphs has already been determined by Brendan McKay and listed in [6], but we shall here also obtain the generating function by degree for the number of these circulant graphs.

A list of symmetric Schur rings over \mathbb{Z}_{27} was obtained using the package COCO (see [3]). (By a “symmetric Schur-Ring”, we mean one in which every basic set T satisfies $-T = T$. This is sufficient for our purpose of enumerating undirected circulant graphs.) The following is the list.

$$\begin{aligned}
\mathfrak{S}_1 &= \langle \underline{0}, \underline{1}, \underline{26}, \underline{2}, \underline{25}, \underline{3}, \underline{24}, \underline{4}, \underline{23}, \underline{5}, \underline{22}, \underline{6}, \underline{21}, \underline{7}, \underline{20}, \underline{8}, \underline{19}, \underline{9}, \underline{18}, \underline{10}, \underline{17}, \underline{11}, \underline{16}, \underline{12}, \underline{15}, \underline{13}, \underline{14} \rangle, \\
\mathfrak{S}_2 &= \langle \underline{0}, \underline{1}, \underline{26}, \underline{8}, \underline{19}, \underline{10}, \underline{17}, \underline{2}, \underline{25}, \underline{7}, \underline{20}, \underline{11}, \underline{16}, \underline{3}, \underline{24}, \underline{4}, \underline{23}, \underline{5}, \underline{22}, \underline{13}, \underline{14}, \underline{6}, \underline{21}, \underline{9}, \underline{18}, \underline{12}, \underline{15} \rangle, \\
\mathfrak{S}_3 &= \langle \underline{0}, \underline{1}, \underline{26}, \underline{2}, \underline{25}, \underline{4}, \underline{23}, \underline{5}, \underline{22}, \underline{7}, \underline{20}, \underline{8}, \underline{19}, \underline{10}, \underline{17}, \underline{11}, \underline{16}, \underline{13}, \underline{14}, \underline{3}, \underline{24}, \underline{6}, \underline{21}, \underline{9}, \underline{18}, \underline{12}, \underline{15} \rangle, \\
\mathfrak{S}_4 &= \langle \underline{0}, \underline{1}, \underline{26}, \underline{8}, \underline{19}, \underline{10}, \underline{17}, \underline{2}, \underline{25}, \underline{7}, \underline{20}, \underline{11}, \underline{16}, \underline{3}, \underline{24}, \underline{6}, \underline{21}, \underline{12}, \underline{15}, \underline{4}, \underline{23}, \underline{5}, \underline{22}, \underline{13}, \underline{14}, \underline{9}, \underline{18} \rangle, \\
\mathfrak{S}_5 &= \langle \underline{0}, \underline{1}, \underline{26}, \underline{2}, \underline{25}, \underline{4}, \underline{23}, \underline{5}, \underline{22}, \underline{7}, \underline{20}, \underline{8}, \underline{19}, \underline{10}, \underline{17}, \underline{11}, \underline{16}, \underline{13}, \underline{14}, \underline{3}, \underline{24}, \underline{6}, \underline{21}, \underline{12}, \underline{15}, \underline{9}, \underline{18} \rangle, \\
\mathfrak{S}_6 &= \langle \underline{0}, \underline{1}, \underline{26}, \underline{2}, \underline{25}, \underline{4}, \underline{23}, \underline{5}, \underline{22}, \underline{7}, \underline{20}, \underline{8}, \underline{19}, \underline{10}, \underline{17}, \underline{11}, \underline{16}, \underline{13}, \underline{14}, \underline{3}, \underline{24}, \underline{6}, \underline{21}, \underline{9}, \underline{18}, \underline{12}, \underline{15} \rangle, \\
\mathfrak{S}_7 &= \langle \underline{0}, \underline{1}, \underline{26}, \underline{2}, \underline{25}, \underline{3}, \underline{24}, \underline{4}, \underline{23}, \underline{5}, \underline{22}, \underline{6}, \underline{21}, \underline{7}, \underline{20}, \underline{8}, \underline{19}, \underline{10}, \underline{17}, \underline{11}, \underline{16}, \underline{12}, \underline{15}, \underline{13}, \underline{14}, \underline{9}, \underline{18} \rangle, \\
\mathfrak{S}_8 &= \langle \underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{10}, \underline{11}, \underline{12}, \underline{13}, \underline{14}, \underline{15}, \underline{16}, \underline{17}, \underline{18}, \underline{19}, \underline{20}, \underline{21}, \underline{22}, \underline{23}, \underline{24}, \underline{25}, \underline{26} \rangle
\end{aligned}$$

In this list, we can observe that \mathfrak{S}_1 is the finest with the smallest automorphism group, while \mathfrak{S}_8 has the largest automorphism group. Therefore \mathfrak{S}_1 contains all the other Schur rings. We may now construct the lattice of Schur rings. This is given in Figure 1.

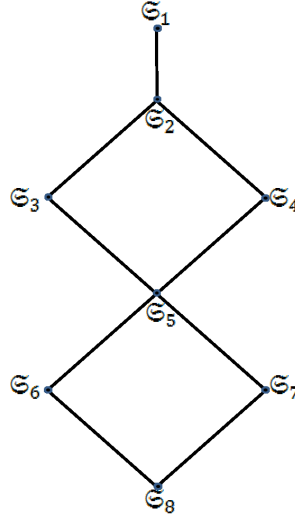


Figure 1. Lattice of all S-rings for $n = 27$ Undirected

Using Equation equation (1.2) we can obtain the generating functions $f_i(t)$.

These are as follows

$$\begin{aligned}
 f_1(t) &= (1 + t^2)^{13} \\
 f_2(t) &= (1 + t^6)^3(1 + t^2)^4 \\
 f_3(t) &= (1 + t^{18})(1 + t^2)^4 \\
 f_4(t) &= (1 + t^6)^4(1 + t^2) \\
 f_5(t) &= (1 + t^{18})(1 + t^6)(1 + t^2) \\
 f_6(t) &= (1 + t^{18})(1 + t^8) \\
 f_7(t) &= (1 + t^{24})(1 + t^2) \\
 f_8(t) &= (1 + t^{26})
 \end{aligned}$$

Table 1 gives a list of the sizes of the automorphism groups and their normalizers. These were again obtained using GAP.

Table 1. Sizes of Automorphism Groups and their Normalizers for the Case $n=27$

G_i	$ G_i $	$ N(G_i) $	$\frac{ G_i }{ N(G_i) }$
G_1	54	486	1/9
G_2	486	4374	1/9
G_3	34992	104976	1/3
G_4	181398528	544195584	1/3
G_5	13060694016	13060694016	1
G_6	286708355039232000	286708355039232000	1
G_7	3656994324480	3656994324480	1
G_8	10888869450418352160768000000	10888869450418352160768000000	1

We may now determine $g_i = g_i(t)$ for $i = 1, 2, \dots, 8$, using equation (1.3) and Figure 1.

$$\begin{aligned}
g_8 &= f_8 = 1 + t^{26} \\
g_7 &= f_7 - g_8 = t^{24} + t^2 \\
g_6 &= f_6 - g_8 = t^{18} + t^8 \\
g_5 &= f_5 - (g_8 + g_7 + g_6) = t^{20} + t^6 \\
g_4 &= \frac{1}{3}(f_4 - g_8 - g_7 - g_6 - g_5) = t^{20} + t^{18} + 2t^{14} + 2t^{12} + t^8 + t^6 \\
g_3 &= \frac{1}{3}(f_3 - g_8 - g_7 - g_6 - g_5) = t^{24} + 2t^{22} + t^{20} + t^6 + 2t^4 + t^2 \\
g_2 &= \frac{1}{9}(f_2 - g_8 - g_7 - g_6 - g_5 - 3g_4 - 3g_3) = t^{18} + 2t^{16} + t^{14} + t^{12} + 2t^{10} + t^8 \\
g_1 &= \frac{1}{9}(f_1 - g_8 - g_7 - g_6 - g_5 - 3g_4 - 3g_3 - 9g_2) \\
&= t^{24} + 8t^{22} + 31t^{20} + 78t^{18} + 141t^{16} + 189t^{14} + 189t^{12} + 141t^{10} + 78t^8 + 31t^6 + \\
&\quad 8t^4 + t^2
\end{aligned}$$

Therefore

$$\begin{aligned}
g(t) &= g_1 + g_2 + \dots + g_8 = t^{26} + 3t^{24} + 10t^{22} + 34t^{20} + 81t^{18} + 143t^{16} + 192t^{14} + \\
&\quad 192t^{12} + 143t^{10} + 81t^8 + 34t^6 + 10t^4 + 3t^2 + 1
\end{aligned}$$

This gives the same generating function as that obtained below using the multiplier method below. It confirms McKay's old result [unpublished, 1995] that there are 928 non-isomorphic, undirected circulant graphs on 27 vertices.

3 The multiplier approach for $n = p^2$ when $p = 3$

Since Ádám's Conjecture does not hold for $n = p^2$ we need the next result which tells us, in terms of their connecting sets, when two circulant graphs of this order are isomorphic. This isomorphism criterion will require us to partition the elements of the connecting sets into *layers*. This is done in the following way: We will first consider the set $\mathbb{Z}'_{p^2} = \mathbb{Z}' - \{0\}$ and divide its elements into two layers, namely Y_0 and Y_1 , where Y_0 will contain those elements which do not have p as a factor and Y_1 will contain those elements which do have p as a factor. A connecting set X is then given by

$$X = X_{(0)} \dot{\cup} X_{(1)}$$

where $X_{(0)} = X \cap Y_0$ and $X_{(1)} = X \cap Y_1$. The layer $X_{(0)}$ is a subset of $\mathbb{Z}_{p^2}^*$ while the layer $X_{(1)}$ is a subset of $p\mathbb{Z}_p^*$. In addition, when these layers are acted upon (multiplicatively) by elements of \mathbb{Z}_n^* , where in this case $n = p^2$, these layers are invariant. In [5] the following isomorphism criterion for circulant graphs of order p^2 was presented.

Theorem 3 ([5]). *Two circulant graphs $\Gamma(\mathbb{Z}_n, X)$ and $\Gamma' = \Gamma(\mathbb{Z}_n, X')$ with $n = p^2$ vertices, are isomorphic if and only if their respective layers are multiplicatively equivalent, i.e.*

$$X'_{(0)} = m_0 X_{(0)}, X'_{(1)} = m_1 X_{(1)}, \tag{M_2}$$

for a pair of multipliers $m_0, m_1 \in \mathbb{Z}_{p^2}^*$. Moreover, in the above, one must have

$$m_0 = m_1 \tag{E}$$

whenever

$$(1 + p)X_{(0)} \neq X_{(0)} \tag{R}$$

We shall illustrate in some detail the use of this result for counting circulants of order 9, based on the treatment given in [5], in order to introduce the use of the inclusion-exclusion principle and also some techniques and notation which will be expanded upon in the next section. Our detailed treatment should help to make the more difficult case of $n = p^3$ clearer.

First of all, in practice, it is easier to count orbits under invariance conditions

$$(1 + p)X_{(0)} = X_{(0)}, \quad (\neg R)$$

that is, when the restrictions have an equality, rather than under the non-invariance condition (R) . Therefore, when the problem under consideration includes the non-invariance condition (R) , this is changed to the invariance condition $(\neg R)$ and then the result is subtracted from the total amount.

Let us consider the case when $n = 9$. In this case we have

$$\begin{aligned} \mathbb{Z}_9^* &= \{1, 2, 4, 5, 7, 8\} \text{ and} \\ \mathbb{Z}_9' &= \{1, 2, 3, 4, 5, 6, 7, 8\} \end{aligned}$$

that is, the connecting set X is a subset of \mathbb{Z}_9' and the multipliers m_0 and m_1 come from \mathbb{Z}_9^* . Let Y_0 and Y_1 be the two layers of \mathbb{Z}_9' . Therefore $\mathbb{Z}_9' = Y_0 \dot{\cup} Y_1$ where

$$\begin{aligned} Y_0 &= \{1, 2, 4, 5, 7, 8\} \\ Y_1 &= \{3, 6\} \end{aligned}$$

The connecting sets are then given by the layers

$$\begin{aligned} X_{(0)} &= X \cap Y_0 \\ X_{(1)} &= X \cap Y_1 \end{aligned}$$

Now two circulant graphs may be isomorphic either under one multiplier, that is, $m_0 = m_1$, or two distinct multipliers, that is, when $m_0 \neq m_1$. From Theorem 3, we have that the non-invariance condition (R) , holds only when the multipliers are equal. Therefore in order to count those circulant graphs which are equivalent via two different multipliers, we need to consider the invariance relation $(\neg R)$ given by $4X_{(0)} = X_{(0)}$. One must note however, that this relation may still hold when the multipliers are equal.

When enumerating under this invariance condition, the set $X_{(0)}$ must be taken from whole subsets of Y_0 which are invariant under $4Y_0 = Y_0$ such that $4X_{(0)} = X_{(0)}$. This will give the partition of Y_0 as Y_0^* . In this case $Y_0^* = \{\{1, 4, 7\}, \{2, 8, 5\}\}$. Therefore under the condition $4X_{(0)} = X_{(0)}$, the

set $X_{(0)}$ must be a union of these parts and the multiplicative action is on the sets $\{1, 4, 7\}$ and $\{2, 8, 5\}$. That is, $X_{(0)}$ must either contain all of the set $\{1, 4, 7\}$, or none of it and similarly all of the set $\{2, 8, 5\}$ or none of it.

In order to count the number of non-isomorphic directed circulant graphs on 9 vertices, we will divide the counting problem into two subproblems $A_1(9)$ and $A_2(9)$, in which

- $A_1(9)$ counts all those circulant digraphs which are distinct under the invariance condition $(\neg R)$ with no restriction on the multipliers;
- $A_2(9)$ counts all those circulant digraphs which are distinct under the non-invariance condition (R) and $m_0 = m_1$.

Let us first consider $A_1(9)$. In this case we have the invariance condition $4X_{(0)} = X_{(0)}$ and no restriction on the multipliers, that is, two circulant graphs may be equivalent under one multiplier or two distinct multipliers.

Here we need the action of $\mathbb{Z}_9^* \times \mathbb{Z}_9^*$ on $\{Y_1 \cup Y_0^*\}$. The multiplier on Y_1 can be the same or different from that on Y_0^* . Therefore we have to consider the action of all $(i, j) \in \mathbb{Z}_9^* \times \mathbb{Z}_9^*$ on $\{3, 6\} \cup \{F, H\}$ where $F = \{1, 4, 7\}$ and $H = \{2, 8, 5\}$. For example $(2, 5)$ has the action $(3 \ 6)(F \ H)$, where 2 acts on $\{3, 6\}$ and 5 acts on $\{F, H\}$. This gives the monomial $x_1^2 x_2$. There are 6^2 such actions since $|\mathbb{Z}_9^*| = 6$. We may however, determine $A_1(9)$ more simply, by finding the cycle index of \mathbb{Z}_9^* on $\{3, 6\}$ and \mathbb{Z}_9^* on $\{F, H\}$ and take the product. Both of these are equivalent to the action of \mathbb{Z}_9^* on $\{1, 2\} \bmod 3$, so we may simply find the latter and square.

Action of	Action	Cycle Structure
1	$(1)(2)$	x_1^2
2	$(1, 2)$	x_2
4	$(1)(2)$	x_1^2
5	$(1, 2)$	x_2
7	$(1)(2)$	x_1^2
8	$(1, 2)$	x_2

Therefore the cycle index is:

$$\frac{1}{6}(3x_1^2 + 3x_2)$$

Squaring and simplifying gives

$$A_2(9) = \frac{1}{4}(x_1^2 + x_2)^2$$

Substituting $x_i = 2$ for all i we obtain $A_1(9) = 9$

Let us now consider $A_2(9)$. Since in this case we have the non-invariance condition (R) , we shall make use of the counting procedure described previously. Let

A_{21} count all the circulants which are distinct under $m_0 = m_1$, that is, when

Ádám's condition holds and

A_{22} count the number of circulants which are distinct under $(\neg R)$ and having

$$m_0 = m_1$$

Then

$$A_2(9) = A_{21} - A_{22}.$$

Let us first determine A_{21} . Here we need to count all circulant graphs assuming Ádám's conjecture holds, that is the number of one-multiplier equivalent directed circulant graphs. This is the number of orbits under the action $(\mathbb{Z}_9^*, \mathbb{Z}'_9)$.

Action of	Action	Cycle Structure
1	(1)(2)(3)(4)(5)(6)(7)(8)	x_1^8
2	(1,2,4,8,7,5)(3,6)	x_2x_6
4	(1,4,7)(2,8,5)(3)(6)	$x_1^2x_3^2$
5	(1,5,7,8,4,2)(3,6)	x_2x_6
7	(1,7,4)(2,5,8)(3)(6)	$x_1^2x_3^2$
8	(1,8)(2,7)(4,5)(3,6)	x_2^4

Therefore the cycle index is given by:

$$A_{21} = \frac{1}{6}(x_1^8 + 2x_2x_6 + 2x_1^2x_3^2 + x_2^4)$$

and substituting $x_i = 2$ for all i we obtain

$$A_{21} = \frac{1}{6}(2^8 + 2(2)(2) + 2(2)^2(2)^2 + (2)^4) = 52$$

Let us now determine A_{22} . Since in A_{22} the multipliers are equal and we have that $4X_0 = X_0$, we need to consider the orbits of the action $(\mathbb{Z}_9^*, Y_0^* \cup Y_1)$, that is, $(\mathbb{Z}_9^*, \{\{1, 4, 7\}, \{2, 8, 5\}, 3, 6\})$. The sets $\{1, 4, 7\}$ and $\{2, 8, 5\}$ are blocks, that is, each block must appear whole as a neighbour or not at all. The contents of X_1 do not influence whether or not X_0 is 4-invariant, therefore 3 and 6 are acted upon separately. Now the action of \mathbb{Z}_9^* on $\{\{1, 4, 7\}, \{2, 8, 5\}\}$ as two points is equivalent to the action of \mathbb{Z}_9^* on $\{1, 2\} \bmod 3$. Therefore, this action may be expressed as $(\mathbb{Z}_9^*, \{1', 2', 1, 2\}) \bmod 3$ where $1', 2'$ represent $\{1, 4, 7\}$ and $\{2, 8, 5\}$ respectively.

Action of	Action	Cycle Structure
1	$(1)(2)(1')(2')$	x_1^4
2	$(1, 2)(1', 2')$	x_2^2
4	$(1)(2)(1')(2')$	x_1^4
5	$(1, 2)(1', 2')$	x_2^2
7	$(1)(2)(1')(2')$	x_1^4
8	$(1, 2)(1', 2')$	x_2^2

Therefore the cycle index is given by:

$$A_{22} = \frac{1}{6}(3x_1^4 + 3x_2^2) = \frac{1}{2}(x_1^4 + x_2^2)$$

and substituting $x_i = 2$ for all i we obtain

$$A_{22} = \frac{1}{2}(2^4 + 2^2) = 10$$

Therefore we have $A_2(9) = A_{21} - A_{22} = 52 - 10 = 42$.

Therefore combining our results we obtain:

$$A_1(9) + A_2(9) = 9 + 42 = 51.$$

This means that 51 directed, non-isomorphic circulant graphs on 9 vertices exist.

Summarizing the above in order to see more clearly the role of inclusion-exclusion, what we have essentially, is the set A_{21} which counts the number of distinct circulant graphs under the conditions (R) and $(\neg R)$ and $m_0 = m_1$.

The set A_1 counts the set of distinct circulants under the condition $(\neg R)$ with no restriction on the multipliers (that is the multipliers could be the same or different). Now what we require is $|A_{21} \cup A_1|$. We know that $|A_{21} \cup A_1| = |A_{21}| + |A_1| - |A_{21} \cap A_1|$, where $|A_{21} \cap A_1|$ counts all those circulant graphs with $(\neg R)$ and $m_1 = m_0$. This is simply the set A_{22} mentioned above.

The same procedure may be repeated for undirected circulants. However, as previously stated, we now need a slight modification on the connecting set, that is, the connecting set is a subset X of \mathbb{Z}'_9 which must have the property $X = -X$, and the multipliers m_0 and m_1 come from \mathbb{Z}_9^* .

Since the elements in the connecting sets are paired by inversion, we partition \mathbb{Z}'_9 as

$$\mathbb{Z}'_9 = \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\}$$

One must note that any connection set we shall work with must have either both elements of a given pair or none. The multiplicative action must therefore be taken on these pairs.

In this case we have

$$Y_0 = \{\{1, 8\}, \{2, 7\}, \{4, 5\}\}$$

$$Y_1 = \{3, 6\}$$

still partitioned into inverse pairs and

$$X_{(0)} = X \cap Y_0 \text{ and } X_{(1)} = X \cap Y_1.$$

We note here that $Y_0^* = \{1, 2, 4, 5, 7, 8\}$, that is the two separate blocks obtained previously in Y_0^* are now merged together so that every element and its inverse are in the same set.

In order to determine A_{21} , again we require the orbits of the action $(\mathbb{Z}_9^*, \mathbb{Z}'_9)$ and therefore we will consider

$$(\mathbb{Z}_9^*, \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\}).$$

Again $\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$ are considered as blocks, that is, we may consider $(\mathbb{Z}_9^*, \{K, L, M, N\})$ where $K = \{1, 8\}$, $L = \{2, 7\}$, $M = \{3, 6\}$, $N =$

$\{4, 5\}$. The cycle index corresponding to this action is given by

$$A_{21} = \frac{1}{6}(2x_1^4 + 4x_3x_1)$$

and substituting $x_i = 2$ for all i , we obtain $A_{11} = 8$.

To determine A_{22} we again require the action $(\mathbb{Z}_9^*, Y_0^* \cup Y_1)$. Therefore we have

$$(\mathbb{Z}_9^*, \{\{1, 2, 4, 5, 7, 8\}, \{3, 6\}\})$$

This action may be seen as $(\mathbb{Z}_9^*, \{M, O\})$ where $O = \{1, 2, 4, 5, 7, 8\}$. The cycle index corresponding to this action is

$$A_{22} = \frac{1}{6}(6)x_1^2 = x_1^2$$

and substituting $x_i = 2$ we obtain $A_{22} = 4$.

Therefore

$$A_2(9) = A_{21} - A_{22} = 4.$$

For $A_1(9)$ we need to multiply the cycle indices of the actions $(\mathbb{Z}_9^*, \{3, 6\}) = (\mathbb{Z}_9^*, \{M\})$ and $(\mathbb{Z}_9^*, \{1, 2, 4, 5, 7, 8\}) = (\mathbb{Z}_9^*, \{O\})$. Both these cycle indices are equivalent to x_1 . Therefore

$$A_1(9) = x_1^2.$$

Substituting, we obtain $A_1(9) = 4$.

Therefore the number of non-isomorphic, undirected circulant graphs on 9 vertices is

$$A_1(9) + A_2(9) = 4 + 4 = 8.$$

4 The multiplier approach for $n = p^3$

4.1 The Main Isomorphism Theorem

We shall first state the general isomorphism theorem for circulant graphs which was proved by Klin and Pöschel in [7]. Theorem 3 which we used for the enumeration of circulants of order p^2 is a special case of this. We then

state the special case of the result of Klin and Pöschel for order p^3 , which will be our main tool.

Theorem 4 ([7],[10]). *Let $n = p^k$ (p an odd prime) and let Γ and Γ' be two p^k -circulants with the connection sets X and X' , respectively. Then Γ and Γ' are isomorphic if and only if their respective layers are multiplicatively equivalent, that is,*

$$X'_{(i)} = m_i X_{(i)}, \quad i = 0, 1, \dots, k-1, \quad (M_k)$$

for an arbitrary set of multipliers $m_0, m_1, \dots, m_{k-1} \in \mathbb{Z}_p^*$ which satisfy the following constraints: whenever the layer $X_{(i)}$ satisfies the non-invariance condition

$$(1 + p^{k-i-j-1})X_{(i)} \neq X_{(i)} \quad (R_{ij})$$

for some $i \in \{0, 1, \dots, k-2\}$ and $j \in \{0, 1, \dots, k-2-i\}$, the successive multipliers m_i, \dots, m_{k-j-1} meet the system of congruences

$$\begin{aligned} m_{i+1} &\equiv m_i \pmod{p^{k-i-j-1}}, \\ m_{i+2} &\equiv m_{i+1} \pmod{p^{k-i-j-2}}, \\ &\vdots \\ m_{k-j-1} &\equiv m_{k-j-2} \pmod{p}. \end{aligned} \quad (E_{ij})$$

For the case when $k = 3$, Theorem 4 translates to the following theorem which we shall refer to as the Main Isomorphism Theorem

Theorem 5 (Main Isomorphism Theorem). *Let $n = p^3$ (p an odd prime) and let Γ and Γ' be two p^3 -circulants with the connection sets X and X' , respectively. Then Γ and Γ' are isomorphic if and only if their respective layers are multiplicatively equivalent, that is,*

$$X'_{(0)} = m_0 X_{(0)}, \quad X'_{(1)} = m_1 X_{(1)} \quad X'_{(2)} = m_2 X_{(2)}, \quad (M_3)$$

for an arbitrary set of multipliers $m_0, m_1, m_2 \in \mathbb{Z}_{p^3}^*$. Moreover, in the above, one must have

$$(i) \quad m_1 \equiv m_0 \pmod{p^2} \text{ and } m_2 \equiv m_1 \pmod{p} \quad (E_{00})$$

whenever

$$(1 + p^2)X_{(0)} \neq X_{(0)}, \quad (R_{00})$$

$$(ii) \quad m_1 \equiv m_0 \pmod{p} \quad (E_{01})$$

whenever

$$(1 + p)X_{(0)} \neq X_{(0)}, \quad (R_{01})$$

$$(iii) \quad m_2 \equiv m_1 \pmod{p} \quad (E_{10})$$

whenever

$$(1 + p)X_{(1)} \neq X_{(1)}. \quad (R_{10})$$

Whereas Theorem 3 for p^2 involved two multipliers and two subcases depending on non-invariance conditions on the layers, the isomorphism theorem for p^k involves k multipliers and $\binom{k}{2}$ cases coming from the non-invariance conditions R_{ij} , making it more difficult to apply in practice for enumeration purposes. And what makes the enumeration problem particularly difficult is not only that there are multipliers for the separate layers of the connection sets, but that, depending on non-invariance conditions, some multipliers must be equal in certain cases. Moreover, the intersection between the conditions makes this case even more difficult. The two cases for p^2 involved three different enumeration problems, as we have seen, and for p^3 , the three non-invariance relations R_{00}, R_{01}, R_{10} below, will break up into five cases which will eventually give eleven enumeration subproblems, as we shall see below.

4.2 Representation of the Main Isomorphism Theorem

Liskovets and Pöschel in [10] manage, for $n = p^3$ and $n = p^4$, to partition the conditions of the Main Theorem into five parts which makes their use in enumeration much easier. These authors take into consideration all combinations of non-invariance conditions (R_{ij}) , together with the remaining invariance conditions

$$(1 + p^{k-i-j-1})X_{(i)} = X_{(i)} \quad (\neg R_{ij})$$

and make use of a number of results, in order to obtain the subproblem list for counting circulants of order p^k , $k \leq 4$. For details of how this list has been generated from the Main Theorem using results from number theory and walks through a rectangular lattice, the reader is referred to [10]. The necessary information required for the enumeration of p^3 circulants is listed in Table 2, which we therefore take to be a rewording of Theorem 5. This has been obtained from Table 1 in [10].

Table 2. The conditions for isomorphism of circulants of order p^3

Subproblem	Non-Invariance Condition	Invariance Condition	Condition on Multipliers
A_1	\emptyset	$\neg R_{01}, \neg R_{10}$	no restriction
A_2	R_{00}	\emptyset	$m_2 = m_1 = m_0$
A_3	R_{01}	$\neg R_{00}, \neg R_{10}$	$m_1 = m_0$
A_4	R_{10}	$\neg R_{01}$	$m_2 = m_1$
A_5	R_{01}, R_{10}	$\neg R_{00}$	$m_2 = m_1$ and $m_1 \equiv m_0 \pmod{p}$

The five subcases A_1 to A_5 shown in Table 2, give conditions on the three multipliers m_0 , m_1 and m_2 for two circulants of order p^3 to be isomorphic. The three multipliers must satisfy at least one of the conditions. For example A_3 means that if the non-invariance condition R_{01} holds, together with the invariance conditions $\neg R_{00}$ and $\neg R_{10}$, then $m_1 = m_0$ but m_2 can be independent.

In addition to this information, we shall also use, without explicit mention, the following observations [10].

$$(R_{ij}) \Rightarrow (R_{ij'}) \text{ whenever } j' \geq j$$

Therefore

$$\neg(R_{ij'}) \Rightarrow \neg(R_{ij})$$

As a result we have that

$$\neg(R_{01}) \Rightarrow \neg(R_{00}) \tag{4.1}$$

In addition,

$$(E_{ij}) \Rightarrow (E_{i'j}) \text{ whenever } i' \geq i$$

Therefore

$$\neg(E_{i'j}) \Rightarrow \neg(E_{ij})$$

that is

$$\neg(E_{10}) \Rightarrow \neg(E_{00}) \quad (4.2)$$

Now, in order to carry out the enumeration, we shall divide the problem into the five enumeration subproblems corresponding to the conditions for isomorphism A_1, A_2, \dots, A_5 in Table 2.

As we explained before, when the subproblem in question includes one or more non-invariance conditions, these are changed to invariance conditions and then the result is subtracted from the total amount. therefore in order to count under a given non-invariance relation R_{ij} , we first

- (i) Determine the count under the action assuming the invariance relation $\neg(R_{ij})$,
- (ii) Determine the count under the action without any (non)-invariance relations,
- (iii) Subtract the result of (i) from (ii).

This procedure is often complicated by having both invariance and non-invariance conditions. For example to count the number of non-isomorphic circulants in case A_3 of Table 2, we

- (i) First count under the conditions $m_1 = m_0, \neg(R_{00}), \neg(R_{10})$.
- (ii) Then count under the conditions $m_1 = m_0, \neg(R_{01}), \neg(R_{00}), \neg(R_{10})$.
- (iii) Subtract the result of (ii) from (i).

Having counted the number of non-isomorphic circulants under each of the five isomorphism conditions we then need to calculate $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$ and therefore we would need to consider their intersections. It however transpires that these intersections are empty. This can be seen by considering the invariance and non-invariance relations. This is the main reason why it is easier to do enumeration using the formulation of Theorem 5 as in Table 2.

We shall illustrate how these five problems lead to eleven subproblems using the case of undirected circulant graphs of order p^k for $p = k = 3$. The

actual values given by these subproblems and their generating functions will be given in the next two subsections for $p = 3$ and $p = 5$, respectively.

So, we have \mathbb{Z}_{27}^* being the set of units in \mathbb{Z}_{27} and \mathbb{Z}_{27}' the set $\mathbb{Z}_{27} - \{0\}$. Therefore the connection set of the circulant graph is a subset X of \mathbb{Z}_{27}' and the multipliers m_0, m_1, m_2 come from \mathbb{Z}_{27}^* .

Let Y_0, Y_1, Y_2 , be the three layers of \mathbb{Z}_{27}' where Y_0 contains all elements of \mathbb{Z}_{27}' which are relatively prime to 27, Y_1 contains those elements which are divisible by 3 and not by 9 and Y_2 contains those divisible by 9.

Now by theorem 5, the non-invariance conditions in this case are:

$$R_{00} : 10X_{(0)} \neq X_{(0)}$$

$$R_{01} : 4X_{(0)} \neq X_{(0)}$$

$$R_{10} : 4X_{(1)} \neq X_{(1)},$$

where $X_{(0)} = X \cap \mathbb{Z}_{27}^*$ that is, $X_{(0)} = X \cap Y_0$ and $X_{(1)} = X \cap 3\mathbb{Z}_9^*$, that is, $X_{(1)} = X \cap Y_1$. Recall that $X_{(2)} = X \cap Y_2$.

As described in the p^2 case, when we enumerate under an invariance condition, such as $10X_{(0)} = X_{(0)}$, we must take $X_{(0)}$ from whole subsets of Y_0 which are invariant under $10Y_0 = Y_0$. These subsets partition Y_0 , therefore under the condition $10X_{(0)} = X_{(0)}$, the set $X_{(0)}$ must be a union of these parts. Therefore the multiplicative action is taken on these parts or blocks.

We shall denote the partitioned set corresponding to the invariance condition $10Y_0 = Y_0$ by Y_0^* , that corresponding to $4Y_0 = Y_0$ by Y_0^{**} , and that corresponding to the invariance condition $4Y_1 = Y_1$ by Y_1^* . We have that

$$Y_0^* = \{\{1, 10, 19\}, \{2, 20, 11\}, \{4, 13, 22\}, \{5, 23, 14\}, \{7, 16, 25\}, \{8, 26, 17\}\}$$

$$Y_0^{**} = \{\{1, 4, 16, 10, 13, 25, 19, 22, 7\}, \{2, 8, 5, 20, 26, 23, 11, 17, 14\}\}$$

$$Y_1^* = \{\{3, 12, 21\}, \{6, 24, 15\}\}$$

Now consider first the subproblem A_1 . In this case we have that, when the invariance conditions $\neg R_{01}$ and $\neg R_{10}$ hold, there is no restriction on the multipliers. Also, A_1 does not include any non-invariance condition, making this subproblem easier because it does not split into further subproblems. We

have here an action on the blocks arising from

$$\begin{aligned} 4X_{(0)} &= X_{(0)} \text{ and} \\ 4X_{(1)} &= X_{(1)} \end{aligned}$$

Under the invariance condition $4X_{(0)} = X_{(0)}$, we must take $X_{(0)}$ from whole subsets of Y_0 which are invariant under $4Y_0 = Y_0$. Therefore the set $X_{(0)}$ must be a union of the blocks in Y_0^{**} . Consequently, instead of Y_0 we shall make use of Y_0^{**} . Similarly, under the invariance condition $4X_{(1)} = X_{(1)}$, we must take $X_{(1)}$ from whole subsets of Y_1 which are invariant under $4Y_1 = Y_1$. These subsets partition Y_1 as Y_1^* . Therefore under the condition $4X_{(1)} = X_{(1)}$, the set $X_{(1)}$ must be a union of these parts. We shall therefore use Y_1^* instead of Y_1 .

Since we have no restriction on the multipliers here, we have the action whose cycle index is the product of the cycle indices $I_{(\mathbb{Z}_{27}^*, Y_0^{**})}$, $I_{(\mathbb{Z}_{27}^*, Y_1^*)}$ and $I_{(\mathbb{Z}_{27}^*, Y_2)}$.

Let us now consider A_2 . Here we have the condition that when $m_0 = m_1 = m_2$ then the non-invariance condition R_{00} must hold. Since in this case we need to consider the non-invariance condition R_{00} , we will use the counting procedure described at the beginning of this chapter. Therefore this isomorphism condition will be split into the following two problems:

A_{21} : The result of our action $(\mathbb{Z}_{27}^*, \mathbb{Z}_{27}')$

A_{22} : The result of an action with $\neg R_{00}$ that is with $10X_{(0)} = X_{(0)}$.

Again since we have the condition $10X_{(0)} = X_{(0)}$ in A_{22} , we must take $X_{(0)}$ from whole subsets of Y_0^* . Since $m_0 = m_1 = m_2$, the action in A_{22} is therefore $(\mathbb{Z}_{27}^*, Y_0^* \cup Y_1 \cup Y_2)$. The required result for A_1 is then given by $A_{21} - A_{22}$.

Let us now consider A_3 . Here we have the conditions R_{01} , $\neg R_{00}$, $\neg R_{10}$ for $m_1 = m_0$. Therefore in this case our isomorphism problem will again be divided into two problems, namely A_{31} and A_{32} , where

A_{31} : is the result of our action with $\neg R_{00}$ and $\neg R_{10}$, that is, with blocks arising from

$$10X_{(0)} = X_{(0)} \text{ and}$$

$$4X_{(1)} = X_{(1)}$$

and therefore we shall need to use Y_0^* instead of Y_0 and Y_1^* instead of Y_1 , and A_{32} : is the result of our action with $\neg R_{00}$, $\neg R_{10}$ and $\neg R_{01}$, that is, with blocks arising from

$$10X_{(0)} = X_{(0)} \text{ and}$$

$$4X_{(1)} = X_{(1)} \text{ and}$$

$$4X_{(0)} = X_{(0)}$$

In this case however, we know that $\neg(R_{01}) \Rightarrow \neg(R_{00})$, therefore the first equation is redundant. Therefore for A_{32} we shall use Y_0^{**} instead of Y_0 and Y_1^* instead of Y_1 . Since $m_1 = m_0$ and m_2 is independent, the cycle index of our action here, is the product of the cycle indices $I_{(\mathbb{Z}_{27}^*, Y_0 \cup Y_1)}$ and $I_{(\mathbb{Z}_{27}^*, Y_2)}$, blocked as required. This means that the cycle indices of A_{31} and A_{32} are:

$$A_{31} : I_{(\mathbb{Z}_{27}^*, Y_0^* \cup Y_1^*)} \times I_{(\mathbb{Z}_{27}^*, Y_2)}$$

$$A_{32} : I_{(\mathbb{Z}_{27}^*, Y_0^{**} \cup Y_1^*)} \times I_{(\mathbb{Z}_{27}^*, Y_2)}$$

and $A_3 = A_{31} - A_{32}$.

We shall now consider A_4 . Here we have the conditions R_{10} and $\neg R_{01}$ when $m_2 = m_1$. Therefore we will now consider A_{41} and A_{42} as follows:

A_{41} : The result of an action with $\neg R_{01}$, that is with blocks arising from $4X_{(0)} = X_{(0)}$. Therefore in this action $X_{(0)}$ must be a union of parts in Y_0^{**} , as such we will use Y_0^{**} instead of Y_0 .

A_{42} : The result of an action with $\neg R_{01}$ and $\neg R_{10}$. This means the set $X_{(0)}$ must be a union of the blocks in Y_0^{**} and $X_{(1)}$ a union of blocks in Y_1^* . The required result will then be $A_{41} - A_{42}$. Since $m_2 = m_1$ while m_0 is independent, we require

$$I_{(\mathbb{Z}_{27}^*, Y_1 \cup Y_2)} \times I_{(\mathbb{Z}_{27}^*, Y_0)},$$

blocked as required. Therefore we have these cycle indices:

$$A_{41} : I_{(\mathbb{Z}_{27}^*, Y_1 \cup Y_2)} \times I_{(\mathbb{Z}_{27}^*, Y_0^{**})}$$

$$A_{42} : I_{(\mathbb{Z}_{27}^*, Y_1^* \cup Y_2)} \times I_{(\mathbb{Z}_{27}^*, Y_0^{**})}.$$

and $A_4 = A_{41} - A_{42}$.

Finally, we consider A_5 . Once again, A_5 will be divided into the problems A_{51} and A_{52} . Although A_{51} is determined in a manner similar to the previous cases, one should be cautious when determining A_{52} , since this time we have two non-invariance conditions. This means that we have to consider the following:

A_{51} : The result of an action with $\neg R_{00}$, that is, we shall use Y_0^* instead of Y_0 ,

A_{52} : The result of an action with $\neg(R_{01} \text{ and } R_{10})$ and $\neg R_{00}$.

Now for A_{52} , by de Morgan's laws, we have that:

$$\begin{aligned} \neg(R_{01} \text{ and } R_{10}) \text{ and } \neg R_{00} &= (\neg R_{01} \text{ or } \neg R_{10}) \text{ and } \neg R_{00} \\ &= (\neg R_{01} \text{ and } \neg R_{00}) \text{ or } (\neg R_{10} \text{ and } \neg R_{00}) \\ |\neg(R_{01} \text{ and } R_{10}) \text{ and } \neg R_{00}| &= |(\neg R_{01} \text{ and } \neg R_{00})| + |(\neg R_{10} \text{ and } \neg R_{00})| - \\ &\quad |(\neg R_{01} \text{ and } \neg R_{10} \text{ and } \neg R_{00})| \end{aligned}$$

Now $\neg(R_{01}) \Rightarrow \neg(R_{00})$, therefore for A_{52} we have:

$$\begin{aligned} |\neg(R_{01} \text{ and } R_{10}) \text{ and } \neg R_{00}| &= |\neg(R_{01})| + |(\neg(R_{10}) \text{ and } \neg(R_{00}))| - \\ &\quad |(\neg(R_{01}) \text{ and } \neg(R_{10}))|. \end{aligned}$$

Therefore we shall split A_{52} into 3 enumeration subproblems, with the first problem enumerating under the condition $\neg(R_{01})$, the second under $\neg(R_{10})$ and $\neg(R_{00})$ and the last under the invariance conditions $\neg(R_{01})$ and $\neg(R_{10})$. These are the subproblems $A_{521}, A_{522}, A_{523}$, respectively, giving that

$$A_5 = A_{51} - A_{521} - A_{522} + A_{523}.$$

Note that we now have two restrictions on the multipliers, an equality with $m_2 = m_1$ and a congruence, $m_1 \equiv m_0 \pmod{3}$. Therefore in this case, we need to define a group G which will act on $\{Y_1 \cup Y_2 \cup Y_3\}$ (blocked as required

according to the given invariance condition), such that:

(1) The same multiplier acts on all $\{Y_1 \cup Y_2 \cup Y_3\}$

(2) Two different multipliers act:

- a on $Y_1 \cup Y_2$
- a' on Y_0

with $a' \equiv a \pmod{3}$.

Now, since $a' \equiv a \pmod{3}$ implies the possibility that $a' \equiv a$, the second possibility includes the first. Therefore we will construct G as follows: G will contain all ordered pairs (a, a') such that $a, a' \in \mathbb{Z}_{27}^*$ and $a' \equiv a \pmod{3}$. Then (a, a') will act as follows:

$$(a, a')(y) = \begin{cases} ay & \text{if } y \in Y_1 \cup Y_2 \\ a'y & \text{if } y \in Y_0 \end{cases}$$

once again, with Y_0 , Y_1 and Y_2 blocked as required. This would give all actions as in (1) and (2) above. One may verify that G is in fact a group since if $(a, a'), (b, b') \in G$ then $(ab, a'b') \in G$.

The general 11-term formula is therefore

$$A = A_1 + A_{21} - A_{22} + A_{31} - A_{32} + A_{41} - A_{42} + A_{51} - A_{521} - A_{522} + A_{523} \quad (*)$$

where A is the number of nonisomorphic circulant graphs we are considering.

The above analysis can be carried out in an analogous way for $p = 5$, and for the directed or undirected way. To generalise a bit our notation, let us define $A[s; p^3]$ for $s = u$, undirected, $s = d$, directed, and $p = 3, 5$ to be the number of undirected/directed circulant graphs on p^3 vertices. We similarly define $A_i[s; p^3]$, $A_{ij}[s; p^3]$ and $A_{ijk}[s; p^3]$ to be the number of undirected/directed circulant graphs on p^3 vertices making up the corresponding intermediate terms A_i , A_{ij} or A_{ijk} , respectively.

We also let $A(t) := A.[s; p^3](t)$ denote generically the generating function by valency of the type of circulant graph under enumeration and the term (intermediate or otherwise) being considered where the coefficient of t^r equals

the number of circulant graphs under discussion having valency (out-valency, in the undirected case) equal to r . This coefficient is sometimes denoted by $A.[s; p^3, r]$. Therefore the terms defined above, counting the circulant graphs regardless of valency are equal to $A.[s; p^3](1)$.

4.3 Results for the cases $p = 3, 5$, directed and undirected

We first give our main results in Table 3 which shows the number of circulant (di)graphs on 27 and 125 vertices.

Table 3. The number of p^3 -circulant graphs and digraphs, $p = 3, 5$

Quantity	Undirected	Directed
n=27	928	3,728,891
n=25	92,233,720,411,499,283	212,676,479,325,586,539,710,725,989,876,778,596

Next, Table 4 gives the values of some of the intermediate terms which, as defined and described in the previous section, together yield the values in Table 3.

Finally, we give all the final generating functions $A[u; 27](t)$, $A[d; 27](t)$, $A[u; 125](t)$ and $A[d; 127](t)$, that is, the generating functions for all the circulants of orders 27 and 125, undirected and directed.

$$A[u; 27](t) = t^{26} + 3t^{24} + 10t^{22} + 34t^{20} + 81t^{18} + 143t^{16} + 192t^{14} + 192t^{12} + 143t^{10} + 81t^8 + 34t^6 + 10t^4 + 3t^2 + 1$$

$$A[d; 27](t) = t^{26} + 3t^{25} + 23t^{24} + 152t^{23} + 844t^{22} + 3662t^{21} + 12814t^{20} + 36548t^{19} + 86837t^{18} + 173593t^{17} + 295172t^{16} + 429240t^{15} + 536646t^{14} + 577821t^{13} + 536646t^{12} + 429240t^{11} + 295172t^{10} + 173593t^9 + 86837t^8 + 36548t^7 + 12814t^6 + 3662t^5 + 844t^4 + 152t^3 + 23t^2 + 3t + 1$$

Table 4. The number of p^3 -circulant graphs and digraphs, $p = 3, 5$: intermediate contributors and totals

Quantity	Undir n=27	Undir n=125	Dir n=27	Dir n=125
A_1	8	27	27	216
A_{21}	944	92233720411833168	3730584	(*)
A_{22}	48	419664	2776	879609512976
$A_2 = A_{21} - A_{22}$	896	92233720411413504	3727808	(**)
A_{31}	16	1272	156	5034768
A_{32}	8	30	30	420
$A_3 = A_{31} - A_{32}$	8	1242	126	5034348
A_{41}	16	1272	156	5034768
A_{42}	8	30	30	420
$A_4 = A_{41} - A_{42}$	8	1242	126	5034348
A_{51}	32	86592	1168	175943379264
A_{521}	16	1680	200	13423440
A_{522}	16	1680	200	13423440
A_{523}	8	36	36	1044
$(A_{52} = A_{521} + A_{522} - A_{523})$	24	3324	364	26845836
$A_5 = A_{51} - A_{521} - A_{522} + A_{523}$	8	83268	804	175916533428
$A = A_1 + A_2 + A_3 + A_4 + A_5$	928	92233720411499283	3728891	(***)

(*) 212676479325586539710726693559689232

(**) 212676479325586539710725813950176256

(***) 212676479325586539710725989876778596

$$\begin{aligned}
A[u; 125](t) = & t^{124} + 3t^{122} + 45t^{120} + 774t^{118} + 11207t^{116} + 129485t^{114} + 1229657t^{112} + \\
& 9835988t^{110} + 67622641t^{108} + 405731843t^{106} + 2150382085t^{104} + \\
& 10165426468t^{102} + 43203077195t^{100} + 166165624857t^{98} + 581579739591t^{96} + \\
& 1861054998416t^{94} + 5466849215583t^{92} + 14792650391699t^{90} + \\
& 36981626382405t^{88} + 85641660162366t^{86} + 184129570236171t^{84} + \\
& 368259138698205t^{82} + 686301123812811t^{80} + 1193567168903172t^{78} + \\
& 1939546652290065t^{76} + 2948110907190899t^{74} + 4195388602819760t^{72} + \\
& 5593851464926268t^{70} + 6992314336461413t^{68} + 8197885767564289t^{66} + \\
& 9017674350331611t^{64} + 9308567065105337t^{62} + 9017674350331611t^{60} + \\
& 8197885767564289t^{58} + 6992314336461413t^{56} + 5593851464926268t^{54} + \\
& 4195388602819760t^{52} + 2948110907190899t^{50} + 1939546652290065t^{48} + \\
& 1193567168903172t^{46} + 686301123812811t^{44} + 368259138698205t^{42} + \\
& 184129570236171t^{40} + 85641660162366t^{38} + 36981626382405t^{36} + \\
& 14792650391699t^{34} + 5466849215583t^{32} + 1861054998416t^{30} + \\
& 581579739591t^{28} + 166165624857t^{26} + 43203077195t^{24} + 10165426468t^{22} + \\
& 2150382085t^{20} + 405731843t^{18} + 67622641t^{16} + 9835988t^{14} + 1229657t^{12} + \\
& 129485t^{10} + 11207t^8 + 774t^6 + 45t^4 + 3t^2 + 1
\end{aligned}$$

$$\begin{aligned}
A[d; 125](t) = & t^{124} + 3t^{123} + 90t^{122} + 3183t^{121} + 94261t^{120} + 2253202t^{119} + 44660526t^{118} + 752765426t^{117} + \\
& 11009026889t^{116} + 141893725177t^{115} + 1631777381270t^{114} + 16911146021617t^{113} + 159246624819695t^{112} + \\
& 1371970915393992t^{111} + 10877769404828584t^{110} + 79770308932154652t^{109} + 543435229633787791t^{108} + \\
& 3452412046870263651t^{107} + 20522671612153800248t^{106} + 114494904782463078927t^{105} + \\
& 601098250109006348605t^{104} + 2976867524344040005968t^{103} + 13937152500343088195264t^{102} + \\
& 61808241523238111266288t^{101} + 260109683076981986458211t^{100} + 1040438732307841539649001t^{99} + \\
& 3961670557633787406854497t^{98} + 14379396838818630509486185t^{97} + 49814339048764832197753486t^{96} + \\
& 164902639609703309488079986t^{95} + 522191692097394743906419238t^{94} + \\
& 1583419969585645756697355513t^{93} + 4601814286608285713826377107t^{92} + \\
& 12829300435392788915059242212t^{91} + 34337245282963060080748456842t^{90} + \\
& 88295773584762135474231228583t^{89} + 218286773584550853413218320189t^{88} + \\
& 519168542579472256031665746576t^{87} + 1188622715905633892156485804100t^{86} + \\
& 2621065476099602847252984385458t^{85} + 5569764136711656142477377352019t^{84} + \\
& 11411224084970222152064822132446t^{83} + 22550752358393534437019574995528t^{82} + \\
& 43003760311355111831092568853122t^{81} + 79166013300449183486752791791953t^{80} + \\
& 140739579200798547810848407179486t^{79} + 241704929497023593576284109748137t^{78} + \\
& 401127329803571069200022148395060t^{77} + 643475091559895257811475523869150t^{76} + \\
& 998042999154123255509578916762251t^{75} + 1497064498731184884738423828738826t^{74} + \\
& 2172211233453091791403201171682487t^{73} + 3049450385424532709259880563769891t^{72} + \\
& 4142649580199365187088253032770042t^{71} + 5446817040632498674709258423750752t^{70} + \\
& 6932312597168634673342968879821758t^{69} + 8541599450082782011722315254853863t^{68} + \\
& 10189978291326827658936469293236257t^{67} + 11771181819291335403215209136236758t^{66} + \\
& 13167762713105561632909721570633733t^{65} + 14265076272531025106827702213046251t^{64} + \\
& 14966637400688288631941104298816390t^{63} + 15208034778118744904852502416921288t^{62} + \\
& +14966637400688288631941104298816390t^{61} + 14265076272531025106827702213046251t^{60} + \\
& 13167762713105561632909721570633733t^{59} + 11771181819291335403215209136236758t^{58} + \\
& 10189978291326827658936469293236257t^{57} + 8541599450082782011722315254853863t^{56} + \\
& 6932312597168634673342968879821758t^{55} + 5446817040632498674709258423750752t^{54} + \\
& 4142649580199365187088253032770042t^{53} + 3049450385424532709259880563769891t^{52} + \\
& 2172211233453091791403201171682487t^{51} + 1497064498731184884738423828738826t^{50} + \\
& 998042999154123255509578916762251t^{49} + 643475091559895257811475523869150t^{48} + \\
& 401127329803571069200022148395060t^{47} + 241704929497023593576284109748137t^{46} + \\
& 140739579200798547810848407179486t^{45} + 79166013300449183486752791791953t^{44} + \\
& 43003760311355111831092568853122t^{43} + 22550752358393534437019574995528t^{42} + \\
& 11411224084970222152064822132446t^{41} + 5569764136711656142477377352019t^{40} + \\
& 2621065476099602847252984385458t^{39} + 1188622715905633892156485804100t^{38} + \\
& 519168542579472256031665746576t^{37} + 218286773584550853413218320189t^{36} + \\
& 88295773584762135474231228583t^{35} + 34337245282963060080748456842t^{34} + \\
& 12829300435392788915059242212t^{33} + 4601814286608285713826377107t^{32} + \\
& 1583419969585645756697355513t^{31} + 522191692097394743906419238t^{30} + \\
& 164902639609703309488079986t^{29} + 49814339048764832197753486t^{28} + 14379396838818630509486185t^{27} + \\
& 3961670557633787406854497t^{26} + 1040438732307841539649001t^{25} + 260109683076981986458211t^{24} + \\
& 61808241523238111266288t^{23} + 13937152500343088195264t^{22} + 2976867524344040005968t^{21} + \\
& 601098250109006348605t^{20} + 114494904782463078927t^{19} + 20522671612153800248t^{18} + \\
& 3452412046870263651t^{17} + 543435229633787791t^{16} + 79770308932154652t^{15} + 10877769404828584t^{14} + \\
& 1371970915393992t^{13} + 159246624819695t^{12} + 16911146021617t^{11} + 1631777381270t^{10} + 141893725177t^9 + \\
& 11009026889t^8 + 752765426t^7 + 44660526t^6 + 2253202t^5 + 94261t^4 + 3183t^3 + 90t^2 + 3t + 1
\end{aligned}$$

Tables 5 to 10 in Appendix B give the generating functions for some of the intermediate terms given in Table 4. Space limitations do not permit us to give the generating functions for all the terms, so we present in the Appendix B only those for which interesting and sometimes surprising relationships are discussed in the next section. However, in order to give a complete result at least for one case, we give in Table 10 in Appendix B, the generating functions of all the intermediate terms appearing in the directed case of order 27.

We also observe that the number of nonisomorphic undirected circulants on 27 vertices is now verified by Matan Zif-Av's brute-force methods as well as that of McKay's from 1995, and by the structural and multiplier approaches described here. The generating functions for these circulant are now also verified by the structural (for the undirected case of order 27) and multiplier approaches and also by Zif-Av's methods.

5 Discussion of results: unexpected patterns and some conjectures

5.1 Numerical observations and some identities

Our tables have been derived in Chapter 4 of [4]. A direct phenomenological analysis of the main and intermediate analytical formulae shown in these tables reveals some hidden patterns that need to be explained in general, either combinatorially, algebraically or analytically. First of all, in the four cases we are studying (undirected / directed, $p = 3$ / $p = 5$) the following three 'coincidences' are observed.

$$A_{31}[s; p^3] = A_{41}[s; p^3] \tag{5.1.1}$$

$$A_{32}[s; p^3] = A_{42}[s; p^3] \tag{5.1.2}$$

$$A_{521}[s; p^3] = A_{522}[s; p^3] \tag{5.1.3}$$

and (as a corollary of the first two)

$$A_3[s; p^3] = A_4[s; p^3] \tag{5.1.4}$$

for $p = 3, 5$ and $s = u, d$ (where $A_4[s; p^3] := A_{41}[s; p^3] - A_{42}[s; p^3]$).

For example $A_{31}[u; 125] = A_{41}[u; 125] = 1272$. Notice that their enumeration formulae are distinct. Moreover, refined by valencies, the corresponding pair of generating functions are also distinct. However, unexpectedly at first sight, the multisets of coefficients in these pairs of polynomials coincide. A more thorough analysis enabled us to reveal a simple pattern. Namely, in all four cases, we observe the following identities:

$$A_{31}[s; p^3](t) \equiv A_{41}[s; p^3](t^p) \pmod{t^{p^3-1}}, \quad (5.1.1^t)$$

$$A_{32}[s; p^3](t) \equiv A_{42}[s; p^3](t^p) \pmod{t^{p^3-1}}, \quad (5.1.2^t)$$

$$A_{522}[s; p^3](t) \equiv A_{521}[s; p^3](t^p) \pmod{t^{p^3-1}}, \quad (5.1.3^t)$$

and most spectacularly, as a corollary of the first two,

$$A_3[s; p^3](t) \equiv A_4[s; p^3](t^p) \pmod{t^{p^3-1}}. \quad (5.1.4^t)$$

In particular, for the latter identity and undirected graphs we observe that their expressions are

$$\begin{aligned} A_4[u; 27](t) &= t^{24} + 2t^{22} + t^{20} + t^6 + 2t^4 + t^2, \\ A_3[u; 27](t) &= t^{20} + t^{18} + 2t^{14} + 2t^{12} + t^8 + t^6 \equiv A_4[u; 27](t^3) \pmod{t^{26}} \\ A_4[u; 125](t) &= t^{122} + 7t^{120} + 22t^{118} + 51t^{116} + 79t^{114} + 94t^{112} + \\ &\quad 79t^{110} + 51t^{108} + 22t^{106} + 7t^{104} + t^{102} + t^{72} + 7t^{70} + \\ &\quad 22t^{68} + 51t^{66} + 79t^{64} + 94t^{62} + 79t^{60} + 51t^{58} + \\ &\quad 22t^{56} + 7t^{54} + t^{52} + t^{22} + 7t^{20} + 22t^{18} + 51t^{16} + \\ &\quad 79t^{14} + 94t^{12} + 79t^{10} + 51t^8 + 22t^6 + 7t^4 + t^2, \\ A_3[u; 125](t) &= t^{114} + t^{112} + t^{110} + 7t^{104} + 7t^{102} + 7t^{100} + 22t^{94} + \\ &\quad 22t^{92} + 22t^{90} + 51t^{84} + 51t^{82} + 51t^{80} + 79t^{74} + \\ &\quad 79t^{72} + 79t^{70} + 94t^{64} + 94t^{62} + 94t^{60} + 79t^{54} + \\ &\quad 79t^{52} + 79t^{50} + 51t^{44} + 51t^{42} + 51t^{40} + 22t^{34} + \\ &\quad 22t^{32} + 22t^{30} + 7t^{24} + 7t^{22} + 7t^{20} + t^{14} + t^{12} + t^{10}, \end{aligned}$$

and therefore $A_3[u; 125](t)$ is equivalent to $A_4[u; 125](t^5) \pmod{t^{124}}$.

Thus, for example, $A_4[u; 125]$ contributes 22 circulant graphs of valency 6 into the overall sum, and the same number of circulant graphs of valency $30 = 5 \times 6$ is contributed by $A_3[u; 125]$.

Notice that the transformation

$$\eta_{p,3} : t \rightarrow t^p \quad \text{modulo } t^{p^3-1}$$

in the ring of polynomials of t over the rationals is periodic of order 3. Thus, $A_4[s; p^3](t) \equiv A_3[s; p^3](t^{p^2}) \pmod{t^{p^3-1}}$, etc. Besides, this operation fixes the terms $d \cdot t^{e(p^2+p+1)}$, $e = 0, 1, \dots, p-1$.

Finally, more hidden identities of the same nature are valid in all four cases: $A_1[s; p^3](t)$ and $A_{523}[s; p^3](t)$ are invariant with respect to $\eta_{p,3}$, that is,

$$A_1[s; p^3](t) \equiv A_1[s; p^3](t^p) \pmod{t^{p^3-1}}, \quad (5.1.5^t)$$

and

$$A_{523}[s; p^3](t) \equiv A_{523}[s; p^3](t^p) \pmod{t^{p^3-1}}, \quad (5.1.6^t)$$

as can be seen from our tables. Of course (5.1.5^t) and (5.1.6^t) make no sense for valency-unspecified circulants ($t = 1$).

5.2 Conjecture

We conjecture that the above identities hold in general.

Conjecture. *Identities (5.1.1^t) – (5.1.3^t), (5.1.5^t) and (5.1.6^t) (and, consequently, identities (5.1.4^t) and (5.1.1) – (5.1.4)) are valid in general for all odd prime p and $s = u, d$.*

If valid in general, this conjecture should have a simple formal analytical proof. For comparison, this is the case for two identities similar to (5.1.5^t) and (5.1.6^t) that are valid for intermediate enumerative polynomials for circulant graphs of prime-squared orders (cf. [9, 5]); see Appendix A. They promise a simple analytical proof and even suggest the existence of a direct bijective proof of combinatorial and/or algebraic nature. One idea to guess such valency-violating bijections in our particular cases is the following: to extract from the relative pairs of polynomials the corresponding terms with the coefficients equal to 1 and to compare their corresponding single graphs. For example, as we see above, $A_4[u; 125](t)$ contributes a unique circulant graph of valency 2, and it corresponds to a certain unique circulant graph of valency $5 \times 2 = 10$ counted in $A_3[u; 125](t)$. Likewise, the unique contributors of valen-

cies 22 and 52 correspond to those of valencies 110 and $12 \equiv 5 \times 52 \pmod{124}$, respectively, in $A_3[u; 125](t)$ (furthermore, by complementarity, 110 may be replaced with 14). Hopefully the structural approach can help here (maybe even within the rather elementarily framework of the Isomorphism Theorem and related results?); in such a case the identities would gain some value for the structural theory of circulant graphs.

Diverse curiously looking formal identities are rather characteristic for the enumerators of circulant graphs of prime or prime-squared orders ([9, 5]); But the present ones, if valid in general, are of a new nature, valency-violating although by a simple rule.

It is interesting to note that while the analysis in Appendix A gives some theoretical justification for our Conjecture which is based on empirical results, the identities for p^3 in the Conjecture have served as a hint to discover and prove the results for p^2 given in Appendix A.

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Appendix A: New Identities for Prime-Squared Circulant Graphs

For a better understanding of the identities discussed in Section 5.1 it will be useful and instructive to consider the prime-squared circulants and new identities for them. They follow easily from the enumerative formulae obtained in [5]. Moreover, a direct 1-to- p correspondence between appropriate intermediate subsets of p^2 -circulants used in the proofs in [5] (see the first paragraph of Section 7.3, p.27) suggests the existence of a transparent bijective proof of these identities (serving as a sample, in the future, for p^3 -circulants). For brevity, we concentrate on directed circulant graphs.

By [5] (in distinct designations), the enumerative generating functions for valency specified circulant digraphs of order p^2 ($p > 2$, prime) satisfies the general equation

$$A[d; p^2](t) = A_1[d; p^2](t) + A_{21}[d; p^2](t) - A_{22}[d; p^2](t) \quad (a1)$$

where the RHS terms are the generating functions for appropriate intermediate types of circulant graphs (similar to ones introduced in the p^3 case) and are calculated via the cycle index of the regular cyclic group $\mathfrak{I}_n(\mathbf{x}) := \frac{1}{n} \sum_{r|n} \phi(r) x_r^{n/r}$, where $\mathbf{x} := \{x_1, x_2, \dots\}$. Namely, consider polynomials

$$\mathfrak{D}(p^2; \mathbf{x}, \mathbf{y}) := \mathfrak{I}_{p-1}(\mathbf{x}) \mathfrak{I}_{p-1}(\mathbf{y}), \quad (a2)$$

$$\mathfrak{B}(p^2; \mathbf{x}, \mathbf{y}) := \mathfrak{I}_{p-1}(\mathbf{xy}), \quad (a3)$$

where $\mathbf{xy} := \{x_1 y_1, x_2 y_2, \dots\}$. Then

$$A_1[d; p^2](t) = \mathfrak{D}(p^2; \mathbf{x}, \mathbf{y})|_{\{x_r := 1+t^r, y_r := 1+t^{pr}\}_{r=1,2,\dots}} \quad (a4)$$

$$A_{22}[d; p^2](t) = \mathfrak{B}(p^2; \mathbf{x}, \mathbf{y})|_{\{x_r := 1+t^r, y_r := 1+t^{pr}\}_{r=1,2,\dots}} \quad (a5)$$

A certain similar formula holds for $A_{21}[d; p^2](t)$ as well but it does not seem to possess any interesting pattern.

Now, let $\mathfrak{F}(\mathbf{x}, \mathbf{y})$ be an arbitrary multivariate polynomial and suppose that it is symmetric with respect to the interchange of variables $\mathbf{x} \leftrightarrow \mathbf{y}$ (that is,

$x_1 \leftrightarrow y_1, x_2 \leftrightarrow y_2, \dots$), i.e. satisfies the identity

$$\mathfrak{F}(\mathbf{x}, \mathbf{y}) = \mathfrak{F}(\mathbf{y}, \mathbf{x}). \quad (a6)$$

Consider the following substitution of variables

$$G(t) := \mathfrak{F}(\mathbf{x}, \mathbf{y})|_{\{x_r := g_r(t), y_r := g_r(t^p)\}_{r=1,2,\dots}} \quad (a7)$$

where $g_r(t), r = 1, 2, \dots$, are arbitrary polynomials and p is an arbitrary positive integer. Then the following polynomial congruence is valid:

$$G(t) \equiv G(t^p) \pmod{t^{p^2-1}}.$$

Indeed, combining the substitution (a7) with $t \rightarrow t^p$ we obtain

$$G(t^p) = \mathfrak{F}(\mathbf{x}, \mathbf{y})|_{\{x_r := g_r(t^p), y_r := g_r(t^{p^2})\}_{r=1,2,\dots}}.$$

Since $t^{p^2} \equiv t$ modulo t^{p^2-1} , we have

$$G(t^p) \equiv \tilde{G}(t) \pmod{t^{p^2-1}}$$

where

$$\tilde{G}(t) = \mathfrak{F}(\mathbf{x}, \mathbf{y})|_{\{x_r := g_r(t^p), y_r := g_r(t)\}_{r=1,2,\dots}}.$$

But $\tilde{G}(t) = G(t)$ due to the symmetry property (a6). \square

The polynomials \mathfrak{D} and \mathfrak{B} defined in (a2) and (a3) are symmetric. Therefore we have the following identities.

Proposition.

$$A_1[d; p^2](t) \equiv A_1[d; p^2](t^p) \pmod{t^{p^2-1}}, \quad (a8)$$

$$A_{22}[d; p^2](t) \equiv A_{22}[d; p^2](t^p) \pmod{t^{p^2-1}}, \quad (a9)$$

For example, for $p = 7$ we have the following polynomials (as calculations

show) and easily verifiable congruences:

$$\begin{aligned}
A_1[d; 49](t) &= t^{48} + t^{47} + 3t^{46} + 4t^{45} + 3t^{44} + t^{43} + t^{42} + t^{41} + t^{40} + \\
&\quad 3t^{39} + 4t^{38} + 3t^{37} + t^{36} + t^{35} + 3t^{34} + 3t^{33} + 9t^{32} + 12t^{31} + \\
&\quad 9t^{30} + 3t^{29} + 3t^{28} + 4t^{27} + 4t^{26} + 12t^{25} + 16t^{24} + 12t^{23} + \\
&\quad 4t^{22} + 4t^{21} + 3t^{20} + 3t^{19} + 9t^{18} + 12t^{17} + 9t^{16} + 3t^{15} + 3t^{14} + \\
&\quad t^{13} + t^{12} + 3t^{11} + 4t^{10} + 3t^9 + t^8 + t^7 + t^6 + t^5 + 3t^4 + 4t^3 + \\
&\quad 3t^2 + t + 1 \\
&\equiv A_1[d; 49](t^7) \pmod{t^{48}}
\end{aligned}$$

$$\begin{aligned}
A_{22}[d; 49](t) &= t^{48} + t^{47} + 3t^{46} + 4t^{45} + 3t^{44} + t^{43} + t^{42} + t^{41} + 6t^{40} + \\
&\quad 15t^{39} + 20t^{38} + 15t^{37} + 6t^{36} + t^{35} + 3t^{34} + 15t^{33} + 39t^{32} + \\
&\quad 50t^{31} + 39t^{30} + 15t^{29} + 3t^{28} + 4t^{27} + 20t^{26} + 50t^{25} + 68t^{24} + \\
&\quad 50t^{23} + 20t^{22} + 4t^{21} + 3t^{20} + 15t^{19} + 39t^{18} + 50t^{17} + 39t^{16} + \\
&\quad 15t^{15} + 3t^{14} + t^{13} + 6t^{12} + 15t^{11} + 20t^{10} + 15t^9 + 6t^8 + t^7 + \\
&\quad t^6 + t^5 + 3t^4 + 4t^3 + 3t^2 + t + 1 \\
&\equiv A_{22}[d; 49](t^7) \pmod{t^{48}}
\end{aligned}$$

For the enumerators $A[u; p^2](t)$ and $A[o; p^2](t)$ of undirected and oriented circulant graphs, respectively, of order p^2 the expressions similar to (a1) hold. Accordingly for them the identities similar to (a8) and (a9) hold. They follow likewise from the next equations, proven, respectively, in [5]:

$$A_1[u; p^2](t) = \mathfrak{D}^*(p^2; \mathbf{x}, \mathbf{y})|_{\{x_r := 1+t^{2r}, y_r := 1+t^{2pr}\}_{r=1,2,\dots}},$$

$$A_{22}[u; p^2](t) = \mathfrak{B}^*(p^2; \mathbf{x}, \mathbf{y})|_{\{x_r := 1+t^{2r}, y_r := 1+t^{2pr}\}_{r=1,2,\dots}},$$

where $\mathfrak{D}^*(p^2; \mathbf{x}, \mathbf{y}) := \mathfrak{I}_{\frac{p-1}{2}}(\mathbf{x})\mathfrak{I}_{\frac{p-1}{2}}(\mathbf{y})$, and $\mathfrak{B}^*(p^2; \mathbf{x}, \mathbf{y}) := \mathfrak{I}_{\frac{p-1}{2}}(\mathbf{xy})$ and in [6]:

$$A_1[o; p^2](t) = \mathfrak{D}(p^2; \mathbf{x}, \mathbf{y})|_{\{x_r := 1, y_r := 1\}_{r \text{ even}}, \{x_r^2 := 1+2t^r, y_r^2 := 1+2t^{pr}\}_{r \text{ odd}}},$$

$$A_{22}[o; p^2](t) = \mathfrak{B}(p^2; \mathbf{x}, \mathbf{y})|_{\{x_r := 1, y_r := 1\}_{r \text{ even}}, \{x_r^2 := 1+2t^r, y_r^2 := 1+2t^{pr}\}_{r \text{ odd}}}.$$

**Appendix B: Tables of Generating Functions for some Intermediate Terms
and for all Intermediate Terms for the Directed $n = 27$ Case**

Table 5. Generating functions for some intermediate terms ($n = 27$)

Term	Generating function
$A_1[u; 27](t)$	$t^{26} + t^{24} + t^{20} + t^{18} + t^8 + t^6 + t^2 + 1$
$A_1[d; 27](t)$	$t^{26} + t^{25} + t^{24} + t^{23} + t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + t^{16} + t^{15} + t^{14} + t^{13} + t^{12} + t^{11} + t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$
$A_{31}[u; 27](t)$	$t^{26} + t^{24} + 2t^{20} + 2t^{18} + 2t^{14} + 2t^{12} + 2t^8 + 2t^6 + t^2 + 1$
$A_{31}[d; 27](t)$	$t^{26} + t^{25} + t^{24} + 2t^{23} + 2t^{22} + 2t^{21} + 6t^{20} + 6t^{19} + 6t^{18} + 10t^{17} + 10t^{16} + 10t^{15} + 14t^{14} + 14t^{13} + 14t^{12} + 10t^{11} + 10t^{10} + 10t^9 + 6t^8 + 6t^7 + 6t^6 + 2t^5 + 2t^4 + 2t^3 + t^2 + t + 1$
$A_{32}[u; 27](t)$	$t^{26} + t^{24} + t^{20} + t^{18} + t^8 + t^6 + t^2 + 1$
$A_{32}[d; 27](t)$	$t^{26} + t^{25} + t^{24} + t^{23} + t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + t^{16} + t^{15} + 2t^{14} + 2t^{13} + 2t^{12} + t^{11} + t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$
$A_{41}[u; 27](t)$	$t^{26} + 2t^{24} + 2t^{22} + 2t^{20} + t^{18} + t^8 + 2t^6 + 2t^4 + 2t^2 + 1$
$A_{41}[d; 27](t)$	$t^{26} + 2t^{25} + 6t^{24} + 10t^{23} + 14t^{22} + 10t^{21} + 6t^{20} + 2t^{19} + t^{18} + t^{17} + 2t^{16} + 6t^{15} + 10t^{14} + 14t^{13} + 10t^{12} + 6t^{11} + 2t^{10} + t^9 + t^8 + 2t^7 + 6t^6 + 10t^5 + 14t^4 + 10t^3 + 6t^2 + 2t + 1$
$A_{42}[u; 27](t)$	$t^{26} + t^{24} + t^{20} + t^{18} + t^8 + t^6 + t^2 + 1$
$A_{42}[d; 27](t)$	$t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + t^{16} + t^{15} + t^{14} + 2t^{13} + t^{12} + t^{11} + t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + 2t^4 + t^3 + t^2 + t + 1$
$A_{521}[u; 27](t)$	$t^{26} + 2t^{24} + 2t^{22} + 2t^{20} + t^{18} + t^8 + 2t^6 + 2t^4 + 2t^2 + 1$
$A_{521}[d; 27](t)$	$t^{26} + 2t^{25} + 6t^{24} + 10t^{23} + 14t^{22} + 10t^{21} + 6t^{20} + 2t^{19} + t^{18} + t^{17} + 4t^{16} + 10t^{15} + 20t^{14} + 26t^{13} + 20t^{12} + 10t^{11} + 4t^{10} + t^9 + t^8 + 2t^7 + 6t^6 + 10t^5 + 14t^4 + 10t^3 + 6t^2 + 2t + 1$
$A_{522}[u; 27](t)$	$t^{26} + t^{24} + 2t^{20} + 2t^{18} + 2t^{14} + 2t^{12} + 2t^8 + 2t^6 + t^2 + 1$
$A_{522}[d; 27](t)$	$t^{26} + t^{25} + t^{24} + 2t^{23} + 4t^{22} + 2t^{21} + 6t^{20} + 10t^{19} + 6t^{18} + 10t^{17} + 20t^{16} + 10t^{15} + 14t^{14} + 26t^{13} + 14t^{12} + 10t^{11} + 20t^{10} + 10t^9 + 6t^8 + 10t^7 + 6t^6 + 2t^5 + 4t^4 + 2t^3 + t^2 + t + 1$
$A_{523}[u; 27](t)$	$t^{26} + t^{24} + t^{20} + t^{18} + t^8 + t^6 + t^2 + 1$
$A_{523}[d; 27](t)$	$t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + 2t^{16} + t^{15} + 2t^{14} + 4t^{13} + 2t^{12} + t^{11} + 2t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + 2 + t^4 + t^3 + t^2 + t + 1$

Table 6. Generating functions for some intermediate terms for order 125
(Part 1)

Term	Generating function
$A_1[u; 125](t)$	$t^{124} + t^{122} + t^{120} + t^{114} + t^{112} + t^{110} + t^{104} + t^{102} + t^{100} + t^{74} + t^{72} + t^{70} + t^{64} + t^{62} + t^{60} + t^{54} + t^{52} + t^{50} + t^{24} + t^{22} + t^{20} + t^{14} + t^{12} + t^{10} + t^4 + t^2 + 1$
$A_1[d; 125](t)$	$t^{124} + t^{123} + 2t^{122} + t^{121} + t^{120} + t^{119} + t^{118} + 2t^{117} + t^{116} + t^{115} + 2t^{114} + 2t^{113} + 4t^{112} + 2t^{111} + 2t^{110} + t^{109} + t^{108} + 2t^{107} + t^{106} + t^{105} + t^{104} + t^{103} + 2t^{102} + t^{101} + t^{100} + t^{99} + t^{98} + 2t^{97} + t^{96} + t^{95} + t^{94} + t^{93} + 2t^{92} + t^{91} + t^{90} + 2t^{89} + 2t^{88} + 4t^{87} + 2t^{86} + 2t^{85} + t^{84} + t^{83} + 2t^{82} + t^{81} + t^{80} + t^{79} + t^{78} + 2t^{77} + t^{76} + t^{75} + 2t^{74} + 2t^{73} + 4t^{72} + 2t^{71} + 2t^{70} + 2t^{69} + 2t^{68} + 4t^{67} + 2t^{66} + 2t^{65} + 4t^{64} + 4t^{63} + 8t^{62} + 4t^{61} + 4t^{60} + 2t^{59} + 2t^{58} + 4t^{57} + 2t^{56} + 2t^{55} + 2t^{54} + 2t^{53} + 4t^{52} + 2t^{51} + 2t^{50} + t^{49} + t^{48} + 2t^{47} + t^{46} + t^{45} + t^{44} + t^{43} + 2t^{42} + t^{41} + t^{40} + 2t^{39} + 2t^{38} + 4t^{37} + 2t^{36} + 2t^{35} + t^{34} + t^{33} + 2t^{32} + t^{31} + t^{30} + t^{29} + t^{28} + 2t^{27} + t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + 2t^{17} + t^{16} + t^{15} + 2t^{14} + 2t^{13} + 4t^{12} + 2t^{11} + 2t^{10} + t^9 + t^8 + 2t^7 + t^6 + t^5 + t^4 + t^3 + 2t^2 + t + 1$
$A_{31}[u; 125](t)$	$t^{124} + t^{122} + t^{120} + 2t^{114} + 2t^{112} + 2t^{110} + 8t^{104} + 8t^{102} + 8t^{100} + 22t^{94} + 22t^{92} + 22t^{90} + 51t^{84} + 51t^{82} + 51t^{80} + 80t^{74} + 80t^{72} + 80t^{70} + 96t^{64} + 96t^{62} + 96t^{60} + 80t^{54} + 80t^{52} + 80t^{50} + 51t^{44} + 51t^{42} + 51t^{40} + 22t^{34} + 22t^{32} + 22t^{30} + 8t^{24} + 8t^{22} + 8t^{20} + 2t^{14} + 2t^{12} + 2t^{10} + t^4 + t^2 + 1$
$A_{31}[d; 125](t)$	$t^{124} + t^{123} + 2t^{122} + t^{121} + t^{120} + 2t^{119} + 2t^{118} + 4t^{117} + 2t^{116} + 2t^{115} + 16t^{114} + 16t^{113} + 32t^{112} + 16t^{111} + 16t^{110} + 102t^{109} + 102t^{108} + 204t^{107} + 102t^{106} + 102t^{105} + 536t^{104} + 536t^{103} + 1072t^{102} + 536t^{101} + 536t^{100} + 2126t^{99} + 2126t^{98} + 4252t^{97} + 2126t^{96} + 2126t^{95} + 6744t^{94} + 6744t^{93} + 13488t^{92} + 6744t^{91} + 6744t^{90} + 17310t^{89} + 17310t^{88} + 34620t^{87} + 17310t^{86} + 17310t^{85} + 36803t^{84} + 36803t^{83} + 73606t^{82} + 36803t^{81} + 36803t^{80} + 65376t^{79} + 65376t^{78} + 130752t^{77} + 65376t^{76} + 65376t^{75} + 98104t^{74} + 98104t^{73} + 196208t^{72} + 98104t^{71} + 98104t^{70} + 124812t^{69} + 124812t^{68} + 249624t^{67} + 124812t^{66} + 124812t^{65} + 135264t^{64} + 135264t^{63} + 270528t^{62} + 135264t^{61} + 135264t^{60} + 124812t^{59} + 124812t^{58} + 249624t^{57} + 124812t^{56} + 124812t^{55} + 98104t^{54} + 98104t^{53} + 196208t^{52} + 98104t^{51} + 98104t^{50} + 65376t^{49} + 65376t^{48} + 130752t^{47} + 65376t^{46} + 65376t^{45} + 36803t^{44} + 36803t^{43} + 73606t^{42} + 36803t^{41} + 36803t^{40} + 17310t^{39} + 17310t^{38} + 34620t^{37} + 17310t^{36} + 17310t^{35} + 6744t^{34} + 6744t^{33} + 13488t^{32} + 6744t^{31} + 6744t^{30} + 2126t^{29} + 2126t^{28} + 4252t^{27} + 2126t^{26} + 2126t^{25} + 536t^{24} + 536t^{23} + 1072t^{22} + 536t^{21} + 536t^{20} + 102t^{19} + 102t^{18} + 204t^{17} + 102t^{16} + 102t^{15} + 16t^{14} + 16t^{13} + 32t^{12} + 16t^{11} + 16t^{10} + 2t^9 + 2t^8 + 4t^7 + 2t^6 + 2t^5 + t^4 + t^3 + 2t^2 + t + 1$
$A_{32}[u; 125](t)$	$t^{124} + t^{122} + t^{120} + t^{114} + t^{112} + t^{110} + t^{104} + t^{102} + t^{100} + t^{74} + t^{72} + t^{70} + 2t^{64} + 2t^{62} + 2t^{60} + t^{54} + t^{52} + t^{50} + t^{24} + t^{22} + t^{20} + t^{14} + t^{12} + t^{10} + t^4 + t^2 + 1$
$A_{32}[d; 125](t)$	$t^{124} + t^{123} + 2t^{122} + t^{121} + t^{120} + t^{119} + t^{118} + 2t^{117} + t^{116} + t^{115} + 2t^{114} + 2t^{113} + 4t^{112} + 2t^{111} + 2t^{110} + t^{109} + t^{108} + 2t^{107} + t^{106} + t^{105} + t^{104} + t^{103} + 2t^{102} + t^{101} + t^{100} + t^{99} + t^{98} + 2t^{97} + t^{96} + t^{95} + 4t^{94} + 4t^{93} + 8t^{92} + 4t^{91} + 4t^{90} + 6t^{89} + 6t^{88} + 12t^{87} + 6t^{86} + 6t^{85} + 4t^{84} + 4t^{83} + 8t^{82} + 4t^{81} + 4t^{80} + t^{79} + t^{78} + 2t^{77} + t^{76} + t^{75} + 2t^{74} + 2t^{73} + 4t^{72} + 2t^{71} + 2t^{70} + 6t^{69} + 6t^{68} + 12t^{67} + 6t^{66} + 6t^{65} + 10t^{64} + 10t^{63} + 20t^{62} + 10t^{61} + 10t^{60} + 6t^{59} + 6t^{58} + 12t^{57} + 6t^{56} + 6t^{55} + 2t^{54} + 2t^{53} + 4t^{52} + 2t^{51} + 2t^{50} + 4t^{49} + t^{48} + 2t^{47} + t^{46} + t^{45} + 4t^{44} + 4t^{43} + 8t^{42} + 4t^{41} + 4t^{40} + 6t^{39} + 6t^{38} + 12t^{37} + 6t^{36} + 6t^{35} + 4t^{34} + 4t^{33} + 8t^{32} + 4t^{31} + 4t^{30} + t^{29} + t^{28} + 2t^{27} + t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + 2t^{17} + t^{16} + t^{15} + 2t^{14} + 2t^{13} + 4t^{12} + 2t^{11} + 2t^{10} + t^9 + t^8 + 2t^7 + t^6 + t^5 + t^4 + t^3 + 2t^2 + t + 1$

Table 7. Generating functions for some intermediate terms for order 125
(Part 2)

Term	Generating function
$A_{41}[u; 125](t)$	$t^{124} + 2t^{122} + 8t^{120} + 22t^{118} + 51t^{116} + 80t^{114} + 96t^{112} + 80t^{110} +$ $51t^{108} + 22t^{106} + 8t^{104} + 2t^{102} + t^{100} + t^{74} + 2t^{72} + 8t^{70} + 22t^{68} + 51t^{66} +$ $80t^{64} + 96t^{62} + 80t^{60} + 51t^{58} + 22t^{56} + 8t^{54} + 2t^{52} + t^{50} + t^{24} + 2t^{22} +$ $8t^{20} + 22t^{18} + 51t^{16} + 80t^{14} + 96t^{12} + 80t^{10} + 51t^8 + 22t^6 + 8t^4 + 2t^2 + 1$
$A_{41}[d; 125](t)$	$t^{124} + 2t^{123} + 16t^{122} + 102t^{121} + 536t^{120} + 2126t^{119} + 6744t^{118} +$ $17310t^{117} + 36803t^{116} + 65376t^{115} + 98104t^{114} + 124812t^{113} + 135264t^{112} +$ $124812t^{111} + 98104t^{110} + 65376t^{109} + 36803t^{108} + 17310t^{107} + 6744t^{106} +$ $2126t^{105} + 536t^{104} + 102t^{103} + 16t^{102} + 2t^{101} + t^{100} + t^{99} + 2t^{98} + 16t^{97} +$ $102t^{96} + 536t^{95} + 2126t^{94} + 6744t^{93} + 17310t^{92} + 36803t^{91} + 65376t^{90} +$ $98104t^{89} + 124812t^{88} + 135264t^{87} + 124812t^{86} + 98104t^{85} + 65376t^{84} +$ $36803t^{83} + 17310t^{82} + 6744t^{81} + 2126t^{80} + 536t^{79} + 102t^{78} + 16t^{77} + 2t^{76} +$ $t^{75} + 2t^{74} + 4t^{73} + 32t^{72} + 204t^{71} + 1072t^{70} + 4252t^{69} + 13488t^{68} + 34620t^{67} +$ $73606t^{66} + 130752t^{65} + 196208t^{64} + 249624t^{63} + 270528t^{62} + 249624t^{61} +$ $196208t^{60} + 130752t^{59} + 73606t^{58} + 34620t^{57} + 13488t^{56} + 4252t^{55} +$ $1072t^{54} + 204t^{53} + 32t^{52} + 4t^{51} + 2t^{50} + t^{49} + 2t^{48} + 16t^{47} + 102t^{46} +$ $536t^{45} + 2126t^{44} + 6744t^{43} + 17310t^{42} + 36803t^{41} + 65376t^{40} + 98104t^{39} +$ $124812t^{38} + 135264t^{37} + 124812t^{36} + 98104t^{35} + 65376t^{34} + 36803t^{33} +$ $17310t^{32} + 6744t^{31} + 2126t^{30} + 536t^{29} + 102t^{28} + 16t^{27} + 2t^{26} + t^{25} + t^{24} +$ $2t^{23} + 16t^{22} + 102t^{21} + 536t^{20} + 2126t^{19} + 6744t^{18} + 17310t^{17} + 36803t^{16} +$ $65376t^{15} + 98104t^{14} + 124812t^{13} + 135264t^{12} + 124812t^{11} + 98104t^{10} +$ $65376t^9 + 36803t^8 + 17310t^7 + 6744t^6 + 2126t^5 + 536t^4 + 102t^3 + 16t^2 + 2t + 1$
$A_{42}[u; 125](t)$	$t^{124} + t^{122} + t^{120} + t^{114} + 2t^{112} + t^{110} + t^{104} + t^{102} + t^{100} + t^{74} + t^{72} + t^{70} +$ $t^{64} + 2t^{62} + t^{60} + t^{54} + t^{52} + t^{50} + t^{24} + t^{22} + t^{20} + t^{14} + 2t^{12} + t^{10} + t^4 + t^2 + 1$
$A_{42}[d; 125](t)$	$t^{124} + t^{123} + 2t^{122} + t^{121} + t^{120} + t^{119} + 4t^{118} + 6t^{117} + 4t^{116} + t^{115} + 2t^{114} +$ $6t^{113} + 10t^{112} + 6t^{111} + 2t^{110} + t^{109} + 4t^{108} + 6t^{107} + 4t^{106} + t^{105} + t^{104} + t^{103} +$ $2t^{102} + t^{101} + t^{100} + t^{99} + t^{98} + 2t^{97} + t^{96} + t^{95} + t^{94} + 4t^{93} + 6t^{92} + 4t^{91} + t^{90} +$ $2t^{89} + 6t^{88} + 10t^{87} + 6t^{86} + 2t^{85} + t^{84} + 4t^{83} + 6t^{82} + 4t^{81} + t^{80} + t^{79} + t^{78} + 2t^{77} +$ $t^{76} + t^{75} + 2t^{74} + 2t^{73} + 4t^{72} + 2t^{71} + 2t^{70} + 2t^{69} + 8t^{68} + 12t^{67} + 8t^{66} + 2t^{65} +$ $4t^{64} + 12t^{63} + 20t^{62} + 12t^{61} + 4t^{60} + 2t^{59} + 8t^{58} + 12t^{57} + 8t^{56} + 2t^{55} + 2t^{54} +$ $2t^{53} + 4t^{52} + 2t^{51} + 2t^{50} + t^{49} + t^{48} + 2t^{47} + t^{46} + t^{45} + t^{44} + 4t^{43} + 6t^{42} + 4t^{41} +$ $t^{40} + 2t^{39} + 6t^{38} + 10t^{37} + 6t^{36} + 2t^{35} + t^{34} + 4t^{33} + 6t^{32} + 4t^{31} + t^{30} + t^{29} + t^{28} +$ $2t^{27} + t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + 4t^{18} + 6t^{17} + 4t^{16} + t^{15} +$ $2t^{14} + 6t^{13} + 10t^{12} + 6t^{11} + 2t^{10} + t^9 + 4t^8 + 6t^7 + 4t^6 + t^5 + t^4 + t^3 + 2t^2 + t + 1$
$A_{521}[u; 125](t)$	$t^{124} + 2t^{122} + 8t^{120} + 22t^{118} + 51t^{116} + 80t^{114} + 96t^{112} + 80t^{110} + 51t^{108} +$ $22t^{106} + 8t^{104} + 2t^{102} + t^{100} + t^{74} + 4t^{72} + 14t^{70} + 44t^{68} + 99t^{66} + 160t^{64} +$ $188t^{62} + 160t^{60} + 99t^{58} + 44t^{56} + 14t^{54} + 4t^{52} + t^{50} + t^{24} + 2t^{22} + 8t^{20} +$ $22t^{18} + 51t^{16} + 80t^{14} + 96t^{12} + 80t^{10} + 51t^8 + 22t^6 + 8t^4 + 2t^2 + 1$

Table 8. Generating functions for some intermediate terms for order 125
(Part 3)

Term	Generating function
$A_{521}[d; 125](t)$	$ \begin{aligned} & t^{124} + 2t^{123} + 16t^{122} + 102t^{121} + 536t^{120} + 2126t^{119} + 6744t^{118} + 17310t^{117} + \\ & 36803t^{116} + 65376t^{115} + 98104t^{114} + 124812t^{113} + 135264t^{112} + 124812t^{111} + \\ & 98104t^{110} + 65376t^{109} + 36803t^{108} + 17310t^{107} + 6744t^{106} + 2126t^{105} + \\ & 536t^{104} + 102t^{103} + 16t^{102} + 2t^{101} + t^{100} + t^{99} + 8t^{98} + 60t^{97} + 408t^{96} + 2126t^{95} + \\ & 8504t^{94} + 26932t^{93} + 69240t^{92} + 147107t^{91} + 261504t^{90} + 392256t^{89} + \\ & 499248t^{88} + 540860t^{87} + 499248t^{86} + 392256t^{85} + 261504t^{84} + 147107t^{83} + \\ & 69240t^{82} + 26932t^{81} + 8504t^{80} + 2126t^{79} + 408t^{78} + 60t^{77} + 8t^{76} + t^{75} + \\ & 2t^{74} + 12t^{73} + 92t^{72} + 612t^{71} + 3196t^{70} + 12756t^{69} + 40420t^{68} + 103860t^{67} + \\ & 220710t^{66} + 392256t^{65} + 588464t^{64} + 748872t^{63} + 811384t^{62} + 748872t^{61} + \\ & 588464t^{60} + 392256t^{59} + 220710t^{58} + 103860t^{57} + 40420t^{56} + 12756t^{55} + \\ & 3196t^{54} + 612t^{53} + 92t^{52} + 12t^{51} + 2t^{50} + t^{49} + 8t^{48} + 60t^{47} + 408t^{46} + 2126t^{45} + \\ & 8504t^{44} + 26932t^{43} + 69240t^{42} + 147107t^{41} + 261504t^{40} + 392256t^{39} + \\ & 499248t^{38} + 540860t^{37} + 499248t^{36} + 392256t^{35} + 261504t^{34} + 147107t^{33} + \\ & 69240t^{32} + 26932t^{31} + 8504t^{30} + 2126t^{29} + 408t^{28} + 60t^{27} + 8t^{26} + t^{25} + t^{24} + \\ & 2t^{23} + 16t^{22} + 102t^{21} + 536t^{20} + 2126t^{19} + 6744t^{18} + 17310t^{17} + 36803t^{16} + \\ & 65376t^{15} + 98104t^{14} + 124812t^{13} + 135264t^{12} + 124812t^{11} + 98104t^{10} + \\ & 65376t^9 + 36803t^8 + 17310t^7 + 6744t^6 + 2126t^5 + 536t^4 + 102t^3 + 16t^2 + 2t + 1 \end{aligned} $
$A_{522}[u; 125](t)$	$ \begin{aligned} & t^{124} + t^{122} + t^{120} + 2t^{114} + 4t^{112} + 2t^{110} + 8t^{104} + 14t^{102} + 8t^{100} + 22t^{94} + \\ & 44t^{92} + 22t^{90} + 51t^{84} + 99t^{82} + 51t^{80} + 80t^{74} + 160t^{72} + 80t^{70} + 96t^{64} + \\ & 188t^{62} + 96t^{60} + 80t^{54} + 160t^{52} + 80t^{50} + 51t^{44} + 99t^{42} + 51t^{40} + 22t^{34} + \\ & 44t^{32} + 22t^{30} + 8t^{24} + 14t^{22} + 8t^{20} + 2t^{14} + 4t^{12} + 2t^{10} + t^4 + t^2 + 1 \end{aligned} $
$A_{522}[d; 125](t)$	$ \begin{aligned} & t^{124} + t^{123} + 2t^{122} + t^{121} + t^{120} + 2t^{119} + 8t^{118} + 12t^{117} + 8t^{116} + 2t^{115} + \\ & 16t^{114} + 60t^{113} + 92t^{112} + 60t^{111} + 16t^{110} + 102t^{109} + 408t^{108} + 612t^{107} + \\ & 408t^{106} + 102t^{105} + 536t^{104} + 2126t^{103} + 3196t^{102} + 2126t^{101} + 536t^{100} + \\ & 2126t^{99} + 8504t^{98} + 12756t^{97} + 8504t^{96} + 2126t^{95} + 6744t^{94} + 26932t^{93} + \\ & 40420t^{92} + 26932t^{91} + 6744t^{90} + 17310t^{89} + 69240t^{88} + 103860t^{87} + \\ & 69240t^{86} + 17310t^{85} + 36803t^{84} + 147107t^{83} + 220710t^{82} + 147107t^{81} + \\ & 36803t^{80} + 65376t^{79} + 261504t^{78} + 392256t^{77} + 261504t^{76} + 65376t^{75} + \\ & 98104t^{74} + 392256t^{73} + 588464t^{72} + 392256t^{71} + 98104t^{70} + 124812t^{69} + \\ & 499248t^{68} + 748872t^{67} + 499248t^{66} + 124812t^{65} + 135264t^{64} + \\ & 540860t^{63} + 811384t^{62} + 540860t^{61} + 135264t^{60} + 124812t^{59} + \\ & 499248t^{58} + 748872t^{57} + 499248t^{56} + 124812t^{55} + 98104t^{54} + 392256t^{53} + \\ & 588464t^{52} + 392256t^{51} + 98104t^{50} + 65376t^{49} + 261504t^{48} + 392256t^{47} + \\ & 261504t^{46} + 65376t^{45} + 36803t^{44} + 147107t^{43} + 220710t^{42} + 147107t^{41} + \\ & 36803t^{40} + 17310t^{39} + 69240t^{38} + 103860t^{37} + 69240t^{36} + 17310t^{35} + \\ & 6744t^{34} + 26932t^{33} + 40420t^{32} + 26932t^{31} + 6744t^{30} + 2126t^{29} + 8504t^{28} + \\ & 12756t^{27} + 8504t^{26} + 2126t^{25} + 536t^{24} + 2126t^{23} + 3196t^{22} + 2126t^{21} + \\ & 536t^{20} + 102t^{19} + 408t^{18} + 612t^{17} + 408t^{16} + 102t^{15} + 16t^{14} + 60t^{13} + \\ & 92t^{12} + 60t^{11} + 16t^{10} + 2t^9 + 8t^8 + 12t^7 + 8t^6 + 2t^5 + t^4 + t^3 + 2t^2 + t + 1 \end{aligned} $

Table 9. Generating functions for some intermediate terms for order 125
(Part 4)

Term	Generating function
$A_{523}[u; 125](t)$	$t^{124} + t^{122} + t^{120} + t^{114} + 2t^{112} + t^{110} + t^{104} + t^{102} + t^{100} + t^{74} + 2t^{72} + t^{70} + 2t^{64} + 4t^{62} + 2t^{60} + t^{54} + 2t^{52} + t^{50} + t^{24} + t^{22} + t^{20} + t^{14} + 2t^{12} + t^{10} + t^4 + t^2 + 1$
$A_{523}[d; 125](t)$	$t^{124} + t^{123} + 2t^{122} + t^{121} + t^{120} + t^{119} + 4t^{118} + 6t^{117} + 4t^{116} + t^{115} + 2t^{114} + 6t^{113} + 10t^{112} + 6t^{111} + 2t^{110} + t^{109} + 4t^{108} + 6t^{107} + 4t^{106} + t^{105} + t^{104} + t^{103} + 2t^{102} + t^{101} + t^{100} + t^{99} + 4t^{98} + 6t^{97} + 4t^{96} + t^{95} + 4t^{94} + 16t^{93} + 24t^{92} + 16t^{91} + 4t^{90} + 6t^{89} + 24t^{88} + 36t^{87} + 24t^{86} + 6t^{85} + 4t^{84} + 16t^{83} + 24t^{82} + 16t^{81} + 4t^{80} + t^{79} + 4t^{78} + 6t^{77} + 4t^{76} + t^{75} + 2t^{74} + 6t^{73} + 10t^{72} + 6t^{71} + 2t^{70} + 6t^{69} + 24t^{68} + 36t^{67} + 24t^{66} + 6t^{65} + 10t^{64} + 36t^{63} + 56t^{62} + 36t^{61} + 10t^{60} + 6t^{59} + 24t^{58} + 36t^{57} + 24t^{56} + 6t^{55} + 2t^{54} + 6t^{53} + 10t^{52} + 6t^{51} + 2t^{50} + t^{49} + 4t^{48} + 6t^{47} + 4t^{46} + t^{45} + 4t^{44} + 16t^{43} + 24t^{42} + 16t^{41} + 4t^{40} + 6t^{39} + 24t^{38} + 36t^{37} + 24t^{36} + 6t^{35} + 4t^{34} + 16t^{33} + 24t^{32} + 16t^{31} + 4t^{30} + t^{29} + 4t^{28} + 6t^{27} + 4t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + 4t^{18} + 6t^{17} + 4t^{16} + t^{15} + 2t^{14} + 6t^{13} + 10t^{12} + 6t^{11} + 2t^{10} + t^9 + 4t^8 + 6t^7 + 4t^6 + t^5 + t^4 + t^3 + 2t^2 + t + 1$

Table 10. Generating functions of all terms in the directed case of order 27

A_1	$t^{26} + t^{25} + t^{24} + t^{23} + t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + t^{16} + t^{15} + t^{14} + t^{13} + t^{12} + t^{11} + t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$
A_{21}	$t^{26} + 3t^{25} + 23t^{24} + 152t^{23} + 850t^{22} + 3680t^{21} + 12850t^{20} + 36606t^{19} + 86919t^{18} + 173701t^{17} + 295311t^{16} + 429388t^{15} + 536810t^{14} + 577996t^{13} + 536810t^{12} + 429388t^{11} + 295311t^{10} + 173701t^9 + 86919t^8 + 36606t^7 + 12850t^6 + 3680t^5 + 850t^4 + 152t^3 + 23t^2 + 3t + 1$
A_{22}	$t^{26} + 2t^{25} + 6t^{24} + 11t^{23} + 22t^{22} + 38t^{21} + 65t^{20} + 92t^{19} + 129t^{18} + 172t^{17} + 214t^{16} + 235t^{15} + 263t^{14} + 276t^{13} + 263t^{12} + 235t^{11} + 214t^{10} + 172t^9 + 129t^8 + 92t^7 + 65t^6 + 38t^5 + 22t^4 + 11t^3 + 6t^2 + 2t + 1$
$A_2 = A_{21} - A_{22}$	$t^{25} + 17t^{24} + 141t^{23} + 828t^{22} + 3642t^{21} + 12785t^{20} + 36514t^{19} + 86790t^{18} + 173529t^{17} + 295097t^{16} + 429153t^{15} + 536547t^{14} + 577720t^{13} + 536547t^{12} + 429153t^{11} + 295097t^{10} + 173529t^9 + 86790t^8 + 36514t^7 + 12785t^6 + 3642t^5 + 828t^4 + 141t^3 + 17t^2 + t$
A_{31}	$t^{26} + t^{25} + t^{24} + 2t^{23} + 2t^{22} + 2t^{21} + 6t^{20} + 6t^{19} + 6t^{18} + 10t^{17} + 10t^{16} + 10t^{15} + 14t^{14} + 14t^{13} + 14t^{12} + 10t^{11} + 10t^{10} + 10t^9 + 6t^8 + 6t^7 + 6t^6 + 2t^5 + 2t^4 + 2t^3 + t^2 + t + 1$
A_{32}	$t^{26} + t^{25} + t^{24} + t^{23} + t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + t^{16} + t^{15} + 2t^{14} + 2t^{13} + 2t^{12} + t^{11} + t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$
$A_3 = A_{31} - A_{32}$	$t^{23} + t^{22} + t^{21} + 5t^{20} + 5t^{19} + 5t^{18} + 9t^{17} + 9t^{16} + 9t^{15} + 12t^{14} + 12t^{13} + 12t^{12} + 9t^{11} + 9t^{10} + 9t^9 + 5t^8 + 5t^7 + 5t^6 + t^5 + t^4 + t^3$
A_{41}	$t^{26} + 2t^{25} + 6t^{24} + 10t^{23} + 14t^{22} + 10t^{21} + 6t^{20} + 2t^{19} + t^{18} + t^{17} + 2t^{16} + 6t^{15} + 10t^{14} + 14t^{13} + 10t^{12} + 6t^{11} + 2t^{10} + t^9 + t^8 + 2t^7 + 6t^6 + 10t^5 + 14t^4 + 10t^3 + 6t^2 + 2t + 1$
A_{42}	$t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + t^{16} + t^{15} + t^{14} + 2t^{13} + t^{12} + t^{11} + t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + 2t^4 + t^3 + t^2 + t + 1$
$A_4 = A_{41} - A_{42}$	$t^{25} + 5t^{24} + 9t^{23} + 12t^{22} + 9t^{21} + 5t^{20} + t^{19} + t^{16} + 5t^{15} + 9t^{14} + 12t^{13} + 9t^{12} + 5t^{11} + t^{10} + t^7 + 5t^6 + 9t^5 + 12t^4 + 9t^3 + 5t^2 + t$
A_{51}	$t^{26} + 2t^{25} + 6t^{24} + 11t^{23} + 18t^{22} + 20t^{21} + 29t^{20} + 38t^{19} + 47t^{18} + 64t^{17} + 86t^{16} + 91t^{15} + 109t^{14} + 124t^{13} + 109t^{12} + 91t^{11} + 86t^{10} + 64t^9 + 47t^8 + 38t^7 + 29t^6 + 20t^5 + 18t^4 + 11t^3 + 6t^2 + 2t + 1$
A_{521}	$t^{26} + 2t^{25} + 6t^{24} + 10t^{23} + 14t^{22} + 10t^{21} + 6t^{20} + 2t^{19} + t^{18} + t^{17} + 4t^{16} + 10t^{15} + 20t^{14} + 26t^{13} + 20t^{12} + 10t^{11} + 4t^{10} + t^9 + t^8 + 2t^7 + 6t^6 + 10t^5 + 14t^4 + 10t^3 + 6t^2 + 2t + 1$
A_{522}	$t^{26} + t^{25} + t^{24} + 2t^{23} + 4t^{22} + 2t^{21} + 6t^{20} + 10t^{19} + 6t^{18} + 10t^{17} + 20t^{16} + 10t^{15} + 14t^{14} + 26t^{13} + 14t^{12} + 10t^{11} + 20t^{10} + 10t^9 + 6t^8 + 10t^7 + 6t^6 + 2t^5 + 4t^4 + 2t^3 + t^2 + t + 1$
A_{523}	$t^{26} + t^{25} + t^{24} + t^{23} + 2t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} + 2t^{16} + t^{15} + 2t^{14} + 4t^{13} + 2t^{12} + t^{11} + 2t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + 2t^4 + t^3 + t^2 + t + 1$
$A_{52} = A_{521} + A_{522} - A_{523}$	$t^{26} + 2t^{25} + 6t^{24} + 11t^{23} + 16t^{22} + 11t^{21} + 11t^{20} + 11t^{19} + 6t^{18} + 10t^{17} + 22t^{16} + 19t^{15} + 32t^{14} + 48t^{13} + 32t^{12} + 19t^{11} + 22t^{10} + 10t^9 + 6t^8 + 11t^7 + 11t^6 + 11t^5 + 16t^4 + 11t^3 + 6t^2 + 2t + 1$
$A_5 = A_{51} - A_{521} - A_{522} + A_{523}$	$2t^{22} + 9t^{21} + 18t^{20} + 27t^{19} + 41t^{18} + 54t^{17} + 64t^{16} + 72t^{15} + 77t^{14} + 76t^{13} + 77t^{12} + 72t^{11} + 64t^{10} + 54t^9 + 41t^8 + 27t^7 + 18t^6 + 9t^5 + 2t^4$
$A = A_1 + A_2 + A_3 + A_4 + A_5$	$t^{26} + 3t^{25} + 23t^{24} + 152t^{23} + 844t^{22} + 3662t^{21} + 12814t^{20} + 36548t^{19} + 86837t^{18} + 173593t^{17} + 295172t^{16} + 429240t^{15} + 536646t^{14} + 577821t^{13} + 536646t^{12} + 429240t^{11} + 295172t^{10} + 173593t^9 + 86837t^8 + 36548t^7 + 12814t^6 + 3662t^5 + 844t^4 + 152t^3 + 23t^2 + 3t + 1$